Common Fixed Point Results of Suzuki-type Rational Z_ψ-contractions

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Abstract: -In this paper, we combine the (α, β)-admissible mappings and the simulation function in this paper to create the generalized version of Suzuki - type rational Z_ψ-contraction mapping. This notion is also employed in the setting of metric spaces to get some common fixed point theorems. Appropriate examples are also provided to validate the results acquired.

Key-Words: Suzuki - type rational, (α, β)-admissible mappings, Z-contraction mapping, metric spaces

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1 Introduction

Samet et al., [1], proposed α-ψ-contractive type mapping and α-admissible mappings. Karapinar and Samet, [2], take the concept further by introducing generalized α-ψ-contractive type mapping broaden the Banach contraction principle, Khojastesh et al., [3], presented simulation function and the notion of Z-contraction with respect to simulation function. Argoubi et al., [4], extend the results of Joonaghany et al., [5]. In this paper, we introduce Suzuki - type rational Z_ψ-contraction.

For more results in rational type contractions and Z-contractions, we refer to the papers in, [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], and references therein.

2 Preliminaries

Throughout this article, the term $\mathbb{N}$ refers to the set of all nonnegative integers. Furthermore, $\mathbb{R}$ represents real numbers, and $\mathbb{R}^+ = [0, \infty)$.

Samet et al., [1], defined the class of α- acceptable mappings in 2012.

Definition 2.1. [1] *A mapping $G : \Omega \to \Omega$ is called α-admissible if for all $\sigma, \delta \in \Omega$ we have*

$$\alpha(\sigma, \delta) \geq 1 \quad \textit{implies} \quad \alpha(G\sigma, G\delta) \geq 1,$$

where $\alpha : \Omega \times \Omega \to [0, \infty)$ is a given function.

Definition 2.2. [1] *Let Ω be a nonempty set, $G, H : \Omega \to \Omega$ and $\alpha, \beta : \Omega \times \Omega \to [0, \infty)$. The two mappings (G, H) is called a pair of (α, β)-admissible mappings, if*

$$\alpha(\sigma, \delta) \geq 1 \quad \textit{and} \quad \beta(\sigma, \delta) \geq 1 \quad \textit{implies}$$

$\alpha(G\sigma, H\delta) \geq 1$ *and* $\beta(H\sigma, G\delta) \geq 1$ *and* $\beta(G\sigma, H\delta) \geq 1$ *and* $\alpha(H\sigma, G\delta) \geq 1$ *for all* $\sigma, \delta \in \Omega$.

Khojasteh et al., [3], introduced the simulation function class in 2015. Furthermore, Argoubi et al., [4], modified the simulation function definition and defined it as follows.

Definition 2.3. [4] *A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ that satisfies the following conditions*

(i) $\zeta(\delta, \sigma) < \sigma - \delta$ *for all* $\sigma, \delta > 0$,

(i) *if $\{\delta_n\}$ and $\{\sigma_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \sigma_n = l \in (0, \infty)$, then*

$$\limsup_{n\to\infty} \zeta(\delta_n, \sigma_n) < 0.$$

Joonaghany et al., [5], proposed a new concept of the ψ-simulation function, and with it, the Z_ψ-contraction in the standard metric space. The concept of the Z_ψ-contraction encompasses several distinct types of contraction, including the Z-contraction defined in, [3].

Denote that $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \,|\, \psi$ is continuous and nondecreasing, and $\psi(s) = 0 \Leftrightarrow s = 0\}$.

Definition 2.4. [5] *We say that $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a ψ-simulation function, if there exists $\psi \in \Psi$ such that*

(ζ_1) $\zeta(\delta, \sigma) < \psi(\sigma) - \psi(\delta)$ *for all* $\sigma, \delta > 0$,

(ζ_2) *if $\{\delta_n\}$ and $\{\sigma_n\}$ are the sequences in $(0, \infty)$ such that $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \sigma_n > 0$ then*

$$\limsup_{n\to\infty} \zeta(\delta_n, \sigma_n) < 0.$$

Let Z_ψ is a collection of all ψ-simulation functions. Take note that "simulation" becomes "simulation function" in sentence, [3].

Example 2.5. [5] *Let* $\psi \in \Psi$

(i) $\zeta_1(\delta, \sigma) = k\psi(\sigma) - \psi(\delta)$ *for all* $\sigma, \delta \in [0, \infty)$, *where* $k \in [0, 1)$.

(ii) $\zeta_2(\delta, \sigma) = \varphi(\psi(\sigma)) - \psi(\delta)$ *for all* $\sigma, \delta \in [0, \infty)$, *where* $\varphi : [0, \infty) \to [0, \infty)$ *so that* $\varphi(0) = 0$ *and for each* $\sigma > 0$, $\varphi(\sigma) < \sigma$,

$$\limsup_{\delta \to \sigma} \varphi(\delta) < \sigma.$$

(ii) $\zeta_3(\delta, \sigma) = \psi(\sigma) - \varphi(\sigma) - \psi(\delta)$ *for all* $\sigma, \delta \in [0, \infty)$, *where* $\varphi : [0, \infty) \to [0, \infty)$ *is a mapping such that, for each* $\sigma > 0$,

$$\liminf_{\delta \to \sigma} \varphi(\delta) > 0.$$

It is clear that $\zeta_1, \zeta_2, \zeta_3 \in Z_\psi$.

Lemma 2.6. [16] *Let* (Ω, d) *be a metric space, and let* $\{\sigma_n\}$ *be a sequence in* Ω *such that*

$$\lim_{n \to \infty} d(\sigma_n, \sigma_{n+1}) = 0.$$

If $\{\sigma_{2n}\}$ numbers is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ and monotone increasing sequences of natural $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$ and $d(\sigma_{2m_k}, \sigma_{2n_k}) \geq \varepsilon$ and

(i) $\lim_{n \to \infty} d(\sigma_{2m_k}, \sigma_{2n_k}) = \varepsilon$,

(ii) $\lim_{n \to \infty} d(\sigma_{2m_k - 1}, \sigma_{2n_k + 1}) = \varepsilon$,

(iii) $\lim_{n \to \infty} d(\sigma_{2m_k}, \sigma_{2n_k + 1}) = \varepsilon$,

(iv) $\lim_{n \to \infty} d(\sigma_{2m_k - 1}, \sigma_{2n_k}) = \varepsilon$.

Motivated by the all above results, we develop the concept of Suzuki-type rational Z_ψ-contraction and demonstrate several typical fixed point results in metric spaces. We also provide an example that supports our primary theorem.

3 Main Result

Now we state our main results.

Definition 3.1. *Let* (Ω, d) *be a metric space. Let* $G, H : \Omega \to \Omega$ *be two mappings. we call the pair* (G, H) *is Suzuki - type rational* Z_ψ*-contraction if for all* $\sigma, \delta \in \Omega$ *and* $L \geq 0$ *such that*

$$\frac{1}{2} \min\{d(\sigma, G\sigma), d(\delta, H\delta)\} \leq d(\sigma, \delta) \quad \textit{implies}$$
$$\zeta(\alpha(\sigma, G\sigma) N(\sigma, \delta), M(\sigma, \delta)) \geq 0, \tag{1}$$

where $\zeta \in Z_\psi$,

$$N(\sigma, \delta) = \beta(\delta, H\delta) d(G\sigma, H\delta)$$

and

$$M(\sigma, \delta) = \max\Big\{d(\sigma, \delta), d(\sigma, G\sigma), d(\delta, H\delta), \frac{A(\sigma, \delta) + B(\sigma, \delta)}{1 + d(\sigma, G\sigma) + d(\delta, H\delta)}, \frac{A(\sigma, \delta) + B(\sigma, \delta)}{1 + d(\sigma, H\delta) + d(\delta, G\sigma)}\Big\} + L \min\{d(\sigma, G\sigma), d(\delta, H\delta), d(\sigma, H\delta), d(\delta, G\sigma)\},$$

which

$$A(\sigma, \delta) = d(\sigma, G\sigma) d(\sigma, H\delta)$$

and

$$B(\sigma, \delta) = d(\delta, H\delta) d(\delta, G\sigma).$$

Theorem 3.2. *Let* (Ω, d) *be a complete metric space, and let* $G, H : \Omega \to \Omega$ *be two mappings and* $\alpha, \beta : \Omega \times \Omega \to [0, \infty)$. *Suppose that the following conditions are satisfied*

(i) (G, H) *is pair of* (α, β)*- admissible mappings*,

(ii) *there exists* $\sigma_0 \in \Omega$ *such that* $\alpha(\sigma_0, G\sigma_0) \geq 1$ *and* $\beta(\sigma_0, H\sigma_0) \geq 1$,

(iii) *the pair* (G, H) *is Suzuki - type rational* Z_ψ*-contraction*,

(iv) *either, G and H are continuous or for every sequence* $\{\sigma_n\}$ *in* Ω *such that* $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ *and* $\beta(\sigma_n, \sigma_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\sigma_n \to \rho$, *we have* $\alpha(\sigma, G\sigma) \geq 1$ *and* $\beta(\sigma, H\sigma) \geq 1$.

Then G and H have a unique common fixed point in Ω.

Proof. By condition (ii), there exists $\sigma_0 \in \Omega$ such that $\alpha(\sigma_0, G\sigma_0) \geq 1$. Define the sequence $\{\sigma_n\}$ in Ω by letting $\sigma_1 \in \Omega$ such that

$$\sigma_1 = G\sigma_0, \sigma_2 = H\sigma_1, \sigma_3 = G\sigma_2, \sigma_4 = H\sigma_3$$

continuing in this manner, we obtain

$$G\sigma_n = \sigma_{n+1} \quad \text{and} \quad H\sigma_{n+1} = \sigma_{n+2}.$$

From (G, H) is a pair of (α, β)-admissible, we have

$$\alpha(\sigma_0, G\sigma_0) = \alpha(\sigma_0, \sigma_1) \geq 1,$$
$$\alpha(G\sigma_0, H\sigma_1) = \alpha(\sigma_1, \sigma_2) \geq 1, \quad \text{and}$$
$$\alpha(H\sigma_1, G\sigma_2) = \alpha(\sigma_2, \sigma_3) \geq 1$$

continuing this process, we get

$$\alpha(\sigma_n, \sigma_{n+1}) \geq 1 \quad \text{for all } n \geq 0.$$

In the same way, we get

$$\beta(\sigma_n, \sigma_{n+1}) \geq 1 \quad \text{for all } n \geq 0.$$

If $\sigma_n = \sigma_{n+1}$ for some $n \in \mathbb{N}$, then $\rho = \sigma_n$ is a common fixed point for G or H. Consequently, we assume that $\sigma_n \neq \sigma_{n+1}$ for all $n \in \mathbb{N}$.
Because

$$\frac{1}{2}\min\{d(\sigma_{2n}, G\sigma_{2n}), d(\sigma_{2n+1}, H\sigma_{2n+1})\} \leq d(\sigma_{2n}, \sigma_{2n+1})$$

from (1), we have

$$\zeta(\alpha(\sigma_{2n}, G\sigma_{2n})N(\sigma_{2n}, \sigma_{2n+1}), M(\sigma_{2n}, \sigma_{2n+1})) \geq 0$$

and

$$\psi(M(\sigma_{2n}, \sigma_{2n+1})) - \psi(\alpha(\sigma_{2n}, G\sigma_{2n})N(\sigma_{2n}, \sigma_{2n+1})) > 0.$$

So,

$$\psi(M(\sigma_{2n}, \sigma_{2n+1})) > \psi(\alpha(\sigma_{2n}, G\sigma_{2n})N(\sigma_{2n}, \sigma_{2n+1}).$$

Because ψ is strictly increasing, we have

$$M(\sigma_{2n}, \sigma_{2n+1}) > \alpha(\sigma_{2n}, G\sigma_{2n})N(\sigma_{2n}, \sigma_{2n+1}), \tag{2}$$

where

$$N(\sigma_{2n}, \sigma_{2n+1}) = \beta(\sigma_{2n+1}, H\sigma_{2n+1})d(\sigma_{2n+1}, \sigma_{2n+2})), \tag{3}$$

and

$$\begin{aligned} &M(\sigma_{2n}, \sigma_{2n+1}) \\ &= \max\Big\{d(\sigma_{2n}, \sigma_{2n+1}), d(\sigma_{2n}, \sigma_{2n+1}), \\ &d(\sigma_{2n+1}, \sigma_{2n+2}), \\ &\frac{A(\sigma_{2n}, \sigma_{2n+1}) + B(\sigma_{2n}, \sigma_{2n+1})}{1 + d(\sigma_{2n}, \sigma_{2n+1}) + d(\sigma_{2n+1}, \sigma_{2n+2})}, \\ &\frac{A(\sigma_{2n}, \sigma_{2n+1}) + B(\sigma_{2n}, \sigma_{2n+1})}{1 + d(\sigma_{2n}, \sigma_{2n+2}) + d(\sigma_{2n+1}, \sigma_{2n+1})}\Big\} \\ &+ L\min\{d(\sigma_{2n}, \sigma_{2n+1}), d(\sigma_{2n+1}, \sigma_{2n+2}), \\ &\quad d(\sigma_{2n}, \sigma_{2n+2}), d(\sigma_{2n+1}, \sigma_{2n+1})\}, \end{aligned} \tag{4}$$

which

$$A(\sigma_{2n}, \sigma_{2n+1}) = d(\sigma_{2n}, \sigma_{2n+1})d(\sigma_{2n}, \sigma_{2n+2}) \tag{5}$$

and

$$B(\sigma_{2n}, \sigma_{2n+1}) = d(\sigma_{2n+1}, \sigma_{2n+2})d(\sigma_{2n+1}, \sigma_{2n+1}) \tag{6}$$

From (4), (5) and (6), we obtain

$$\begin{aligned} &M(\sigma_{2n}, \sigma_{2n+1}) \\ &= \max\Big\{d(\sigma_{2n}, \sigma_{2n+1}), d(\sigma_{2n+1}, \sigma_{2n+2}), \\ &\frac{d(\sigma_{2n}, \sigma_{2n+1})d(\sigma_{2n}, \sigma_{2n+2})}{1 + d(\sigma_{2n}, \sigma_{2n+1}) + d(\sigma_{2n+1}, \sigma_{2n+2})}, \\ &\frac{d(\sigma_{2n}, \sigma_{2n+1})d(\sigma_{2n}, \sigma_{2n+2})}{1 + d(\sigma_{2n}, \sigma_{2n+2})}\Big\} \\ &+ L\min\{d(\sigma_{2n}, \sigma_{2n+1}), d(\sigma_{2n+1}, \sigma_{2n+2}), \\ &d(\sigma_{2n}, \sigma_{2n+2}), 0\} \\ &= \max\Big\{d(\sigma_{2n}, \sigma_{2n+1}), d(\sigma_{2n+1}, \sigma_{2n+2})\Big\}. \end{aligned}$$

If $M(\sigma_{2n}, \sigma_{2n+1}) = d(\sigma_{2n+1}, \sigma_{2n+2})$, then by (2) becomes

$$d(\sigma_{2n+1}, \sigma_{2n+2}) < d(\sigma_{2n+1}, \sigma_{2n+2}),$$

which is a contradiction. Thus we conclude that

$$M(\sigma_{2n}, \sigma_{2n+1}) = d(\sigma_{2n}, \sigma_{2n+1}). \tag{7}$$

By (2), we get

$$d(\sigma_{2n+1}, \sigma_{2n+2}) < d(\sigma_{2n}, \sigma_{2n+1}).$$

As a result, we can conclude that the sequence $\{d(\sigma_n, \sigma_{n+1})\}$ is nonnegative and nonincreasing. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(\sigma_n, \sigma_{n+1}) = r.$$

We assert that $r = 0$. Assume, on the other hand, that $r > 0$.

$$\lim_{n\to\infty} d(\sigma_n, \sigma_{n+1}) = \lim_{n\to\infty} M(\sigma_n, \sigma_{n+1}) = r. \tag{8}$$

For each $n \geq 0$ we have

$$\frac{1}{2}\min\{d(\sigma_{2n}, G\sigma_{2n}), d(\sigma_{2n+1}, H\sigma_{2n+1})\} \leq d(\sigma_{2n}, \sigma_{2n+1})$$

from (1), we have

$$\zeta(\alpha(\sigma_{2n}, G\sigma_{2n})N(\sigma_{2n}, \sigma_{2n+1}), M(\sigma_{2n}, \sigma_{2n+1})) \geq 0,$$

where

$$N(\sigma_{2n}, \sigma_{2n+1}) = \beta(\sigma_{2n+1}, H\sigma_{2n+1})d(G\sigma_{2n}, H\sigma_{2n+1})$$

and hence

$$\limsup_{n\to\infty} \zeta(\alpha(\sigma_{2n}, G\sigma_{2n})N(\sigma_{2n}, \sigma_{2n+1}), M(\sigma_{2n}, \sigma_{2n+1})) \geq 0.$$

By condition (ζ_2) of Definition 2.4, we have

$$\limsup_{n\to\infty} \zeta(\alpha(\sigma_{2n}, G\sigma_{2n})N(\sigma_{2n}, \sigma_{2n+1}), M(\sigma_{2n}, \sigma_{2n+1})) < 0,$$

which is a contradiction. Thus we conclude that

$$\lim_{n\to\infty} d(\sigma_n, \sigma_{n+1}) = \lim_{n\to\infty} M(\sigma_n, \sigma_{n+1}) = 0. \quad (9)$$

Now we will demonstrate that $\{\sigma_n\}$ is a Cauchy sequence. Assume, on the other hand, that $\{\sigma_n\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon_0 > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$ and $d(\sigma_{2m_k}, \sigma_{2n_k}) \geq 0$ and

(i) $\lim_{n\to\infty} d(\sigma_{2m_k}, \sigma_{2n_k}) = \varepsilon_0$,

(ii) $\lim_{n\to\infty} d(\sigma_{2m_k-1}, \sigma_{2n_k+1}) = \varepsilon_0$,

(iii) $\lim_{n\to\infty} d(\sigma_{2m_k}, \sigma_{2n_k+1}) = \varepsilon_0$,

(iv) $\lim_{n\to\infty} d(\sigma_{2m_k-1}, \sigma_{2n_k}) = \varepsilon_0$.

As a result of the definition of $M(\sigma, \delta)$, we have

$$\begin{aligned}&\lim_{n\to\infty} M(\sigma_{2n_k}, \sigma_{2m_k-1})\\ &= \lim_{n\to\infty}\Bigg(\max\Bigg\{d(\sigma_{2n_k}, \sigma_{2m_k-1}), d(\sigma_{2n_k}, \sigma_{2n_k+1}),\\ &d(\sigma_{2m_k-1}, \sigma_{2m_k}),\\ &\frac{A(\sigma_{2n_k}, \sigma_{2m_k-1}) + B(\sigma_{2n_k}, \sigma_{2m_k-1})}{1 + d(\sigma_{2n_k}, \sigma_{2n_k+1}) + d(\sigma_{2m_k-1}, \sigma_{2m_k})},\\ &\frac{A(\sigma_{2n_k}, \sigma_{2m_k-1}) + B(\sigma_{2n_k}, \sigma_{2m_k-1}))}{1 + d(\sigma_{2n_k}, \sigma_{2m_k}) + d(\sigma_{2m_k-1}, \sigma_{2n_k+1})}\Bigg\}\\ &+ L\min\{d(\sigma_{2n_k}, \sigma_{2n_k+1}), d(\sigma_{2m_k-1}, \sigma_{2m_k}),\\ &d(\sigma_{2n_k}, \sigma_{2m_k}), d(\sigma_{2m_k-1}, \sigma_{2n_k+1})\}\Bigg),\end{aligned} \quad (10)$$

which

$$A(\sigma_{2n_k}, \sigma_{2m_k-1}) = d(\sigma_{2n_k}, \sigma_{2n_k+1})d(\sigma_{2n_k}, \sigma_{2m_k}) \quad (11)$$

and

$$B(\sigma_{2n_k}, \sigma_{2m_k-1}) = d(\sigma_{2m_k-1}, \sigma_{2m_k})d(\sigma_{2m_k-1}, \sigma_{2n_k+1}). \quad (12)$$

From (10), (11) and (12), we obtain

$$\begin{aligned}&\lim_{n\to\infty} M(\sigma_{2n_k}, \sigma_{2m_k-1})\\ &= \max\{\varepsilon_0, 0, 0, 0, 0\} + L\min\{0, 0, \varepsilon_0, \varepsilon_0\}\\ &= \varepsilon_0\end{aligned} \quad (13)$$

and hence

$$\lim_{k\to\infty} d(\sigma_{2n_k}, \sigma_{2n_k+1}) = \lim_{k\to\infty} M(\sigma_{2n_k}, \sigma_{2m_k-1}) = \varepsilon_0 > 0.$$

By condition (ζ_2) of Definition 2.4, we have

$$\lim_{k\to\infty} d(\sigma_{2n_k}, \sigma_{2n_k+1}) = \lim_{k\to\infty} M(\sigma_{2n_k}, \sigma_{2m_k-1}) = \varepsilon_0 > 0. \quad (14)$$

In contrast, we assert that for sufficiently large $k \in \mathbb{N}$, if $n_k > m_k > k$, then

$$\frac{1}{2}\min\{d(\sigma_{n_k}, G\sigma_{n_k}), d(\sigma_{m_k-1}, H\sigma_{m_k-1})\} > d(\sigma_{n_k}, \sigma_{m_k-1}). \quad (15)$$

When we let as $k \to \infty$ in (15), we get the $\varepsilon_0 \leq 0$, contradiction. Therefore,

$$\frac{1}{2}\min\{d(\sigma_{n_k}, G\sigma_{n_k}), d(\sigma_{m_k-1}, H\sigma_{m_k-1})\} \leq d(\sigma_{n_k}, \sigma_{m_k-1})$$

and from (1), we have

$$\zeta(\alpha(\sigma_{2n_k}, G\sigma_{2n_k})N(\sigma_{2n_k}, \sigma_{2n_k-1}), M(\sigma_{2n_k}, \sigma_{2m_k-1})) \geq 0,$$

where

$$N(\sigma_{2n_k}, \sigma_{2n_k-1}) = \beta(\sigma_{2m_k-1}, H\sigma_{2m_k-1})d(G\sigma_{2n_k}, H\sigma_{2m_k-1}).$$

Therefore,

$$\limsup_{n\to\infty} \zeta(\alpha(\sigma_{2n_k}, G\sigma_{2n_k})N(\sigma_{2n_k}, \sigma_{2n_k-1}), M(\sigma_{2n_k}, \sigma_{2m_k-1})) \geq 0, \quad (16)$$

which contradicts (14). This contradiction proves that $\{\sigma_n\}$ is a Cauchy sequence, and since Ω is complete, there exists $\rho \in \Omega$ such that $\{\sigma_n\} \to \rho$ as $n \to \infty$. We assert that ρ is a fixed point shared by G and H. Because G and H are continuous, we can conclude that

$$\rho = \lim_{n\to\infty} \sigma_{2n+1} = \lim_{n\to\infty} G\sigma_{2n} = G\left(\lim_{n\to\infty} \sigma_{2n}\right) = G\rho$$

and

$$\rho = \lim_{n\to\infty} \sigma_{2n+2} = \lim_{n\to\infty} H\sigma_{2n+1} = H\left(\lim_{n\to\infty} \sigma_{2n+1}\right) = H\rho.$$

Hence, $G\rho = H\rho = \rho$, that is, ρ is a common fixed point of G and H. From (iv), we have for every sequence $\{\sigma_n\}$ in Ω such that $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ and

$\beta(\sigma_n, \sigma_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\sigma_n \to \rho$ as $n \to \infty$, this implies $\sigma_{2n_k+1} \to \rho$ and $\sigma_{2n_k+2} \to \rho$ as $k \to \infty$. Now we show that $G\rho = H\rho = \rho$. Assume $\rho \neq H\rho$. Now we assert that, for each $n \geq 1$, at least one of the following statements is true.

$$\frac{1}{2} d(\sigma_{n_k-1}, \sigma_{n_k}) \leq d(\sigma_{n_k-1}, \rho)$$

or

$$\frac{1}{2} d(\sigma_{n_k}, \sigma_{n_k+1}) \leq d(\sigma_{n_k}, \rho).$$

Assume, on the other hand,

$$\frac{1}{2} d(\sigma_{n_k-1}, \sigma_{n_k}) > d(\sigma_{n_k-1}, \rho)$$

and

$$\frac{1}{2} d(\sigma_{n_k}, \sigma_{n_k+1}) > d(\sigma_{n_k}, \rho).$$

For some $n \geq 1$, we have

$$\begin{aligned} d(\sigma_{n_k-1}, \sigma_{n_k}) &\leq d(\sigma_{n_k-1}, \rho) + d(\rho, \sigma_{n_k}) \\ &< \frac{1}{2}[d(\sigma_{n_k-1}, \sigma_{n_k}) + d(\sigma_{n_k}, \sigma_{n_k+1})] \\ &\leq d(\sigma_{n_k-1}, \sigma_{n_k}), \end{aligned}$$

which is a contradiction, and thus the claim is true. From (1), we have

$$\frac{1}{2} \min\{d(\sigma_{2n_k}, G\sigma_{2n_k}), d(\rho, H\rho)\} \leq d(\sigma_{2n_k}, \rho)$$

implies

$$\begin{aligned} 0 &\leq \zeta(\alpha(\sigma_{2n_k}, G\sigma_{2n_k}) N(\sigma_{2n_k}, \rho), M(\sigma_{2n_k}, \rho)) \\ &< \psi(M(\sigma_{2n_k}, \rho)) - \psi(\alpha(\sigma_{2n_k}, G\sigma_{2n_k}) N(\sigma_{2n_k}, \rho)). \end{aligned}$$

So,

$$\psi(M(\sigma_{2k}, \rho)) > \psi(\alpha(\sigma_{2k}, G\sigma_{2k}) N(\sigma_{2n_k}, \rho)).$$

Because ψ is strictly increasing, we have

$$M(\sigma_{2n_k}, \rho) > \alpha(\sigma_{2n_k}, G\sigma_{2n_k}) N(\sigma_{2n_k}, \rho), \quad (17)$$

where

$$N(\sigma_{2n_k}, \rho) = \beta(\rho, H\rho) d(G\sigma_{2n_k}, H\rho) \quad (18)$$

and

$$\begin{aligned} &M(\sigma_{2n_k}, \rho) \\ &= \max\Big\{ d(\sigma_{2n_k}, \rho), d(\sigma_{2n_k}, G\sigma_{2n_k}), d(\rho, H\rho), \\ &\frac{A(\sigma_{2n_k}, \rho) + B(\sigma_{2n_k}, \rho)}{1 + d(\sigma_{2n_k}, G\sigma_{2n_k}) + d(\rho, H\rho)}, \\ &\frac{A(\sigma_{2n_k}, \rho) + B(\sigma_{2n_k}, \rho)}{1 + d(\sigma_{2n_k}, H\rho) + d(\rho, G\sigma_{2n_k})} \Big\} \\ &+ L \min\{d(\sigma_{2n_k}, G\sigma_{2n_k}), d(\rho, H\rho), d(\sigma_{2n_k}, H\rho), \\ &\quad d(\rho, G\sigma_{2n_k})\}, \end{aligned} \quad (19)$$

which

$$A(\sigma_{2n_k}, \rho) = d(\sigma_{2n_k}, G\sigma_{2n_k}) d(\sigma_{2n_k}, H\rho) \quad (20)$$

and

$$B(\sigma_{2n_k}, \rho) = d(\rho, H\rho) d(\rho, G\sigma_{2n_k}). \quad (21)$$

Letting $k \to \infty$ in (19), we obtain

$$\lim_{k \to \infty} M(\sigma_{2k}, \rho) = d(\rho, H\rho).$$

From (17), we have

$$\begin{aligned} &d(G\sigma_{2n_k}, H\rho) \\ &\leq \alpha(\sigma_{2n_k}, G\sigma_{2n_k}) N(\sigma_{2n_k}, \rho) \\ &< M(\sigma_{2n_k}, \rho), \end{aligned} \quad (22)$$

where

$$N(\sigma_{2n_k}, \rho) = \beta(\rho, H\rho) d(G\sigma_{2n_k}, H\rho).$$

Letting $k \to \infty$ in (22), we obtain

$$d(\rho, H\rho) < d(\rho, H\rho),$$

which is a contradiction. Therefore, $\rho = H\rho$. In the same way, we can find that $\rho = G\rho$. Therefore, the pair (G, H) has a common fixed point $\rho = G\rho = H\rho$.

We claim G and H have a unique common fixed points $\rho, \rho^* \in \Omega$. Therefore $G\rho = H\rho = \rho$, $G\rho^* = H\rho^* = \rho^*$ and $d(\rho, \rho^*) > 0$. Therefore,

$$\begin{aligned} &\frac{1}{2} \min\{d(\rho, G\rho), d(\rho^*, H\rho^*)\} \\ &= \frac{1}{2} \min\{0, 0\} \\ &< d(\rho, \rho^*) \end{aligned}$$

and from (1), we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(\rho, G\rho) N(\rho, \rho^*), M(\rho, \rho^*)) \\ &< \psi(M(\rho, \rho^*)) - \psi(\alpha(\rho, G\rho) N(\rho, \rho^*)). \end{aligned}$$

Because ψ is strictly increasing,

$$d(\rho, \rho^*) < \alpha(\rho, G\rho) N(\rho, \rho^*) < M(\rho, \rho^*), \quad (23)$$

where

$$N(\rho, \rho^*) = \beta(\rho^*, H\rho^*) d(G\rho, H\rho^*)$$

and

$$\begin{aligned} &M(\rho, \rho^*) \\ &= \max\Big\{ d(\rho, \rho^*), d(\rho, G\rho), d(\rho^*, H\rho^*), \\ &\frac{A(\rho, \rho^*) + B(\rho, \rho^*)}{1 + d(\rho, G\rho) + d(\rho^*, H\rho^*)}, \\ &\frac{A(\rho, \rho^*) + B(\rho, \rho^*)}{1 + d(\rho, H\rho^*) + d(\rho^*, G\rho)} \Big\} \\ &+ L \min\{d(\rho, G\rho^*), d(\rho^*, H\rho^*), d(\rho, H\rho^*), \\ &\quad d(\rho^*, G\rho)\}, \end{aligned} \quad (24)$$

which

$$A(\rho,\rho^*) = d(\rho, G\rho)d(\rho, H\rho^*) \tag{25}$$

and

$$B(\rho,\rho^*) = d(\rho^*, H\rho^*)d(\rho^*, G\rho). \tag{26}$$

From (24), (25) and (26), we obtain

$$M(\rho,\rho^*) = d(\rho,\rho^*) > 0. \tag{27}$$

From (23) and (27), we have

$$\begin{aligned} d(\rho,\rho^*) &< \alpha(\rho, G\rho)\beta(\rho^*, H\rho^*)d(\rho,\rho^*) \\ &< M(\rho,\rho^*) \\ &= d(\rho,\rho^*), \end{aligned}$$

which is a contradiction. Therefore, G and H have a unique common fixed point. ☐

Corollary 3.3. *Let (Ω, d) be a complete metric space, and let $G : \Omega \to \Omega$ be a mapping and $\alpha, \beta : \Omega \times \Omega \to [0,\infty)$. Assume that the following conditions are satisfied*

(i) *if for all $\sigma, \delta \in \Omega$,*

$$\frac{1}{2}\min\{d(\sigma, G\sigma), d(\delta, G\delta)\} \le d(\sigma,\delta) \quad \textit{implies} \quad \zeta(\alpha(\sigma, G\sigma)N(\sigma,\delta), M(\sigma,\delta)) \ge 0, \tag{28}$$

where $\zeta \in Z_\psi$,

$$N(\sigma,\delta) = \beta(\delta, H\delta)d(G\sigma, H\delta)$$

and

$$\begin{aligned} &M(\sigma,\delta) \\ &= \max\Big\{d(\sigma,\delta), d(\sigma, G\sigma), d(\delta, G\delta), \\ &\frac{d(\sigma, G\sigma)d(\sigma, G\delta) + d(\delta, G\delta)d(\delta, G\sigma)}{1 + d(\sigma, G\sigma) + d(\delta, G\delta)}, \\ &\frac{d(\sigma, G\sigma)d(\sigma, G\delta) + d(\delta, G\delta)d(\delta, G\sigma)}{1 + d(\sigma, G\delta) + d(\delta, G\sigma)}\Big\} \\ &+ L\min\{d(\sigma, G\sigma), d(\delta, G\delta), d(\sigma, G\delta), d(\delta, G\sigma)\}, \end{aligned}$$

(ii) *G is (α,β) admissible mapping,*

(iii) *there exists $\sigma_0 \in \Omega$ such that $\alpha(\sigma_0, G\sigma_0) \ge 1$,*

(iv) *either, G and H are continuous or for every sequence $\{\sigma_n\}$ in Ω such that $\alpha(\sigma_n, \sigma_{n+1}) \ge 1$ and $\beta(\sigma_n, \sigma_{n+1}) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $\sigma_n \to \rho$, we have $\alpha(\sigma, G\sigma) \ge 1$ and $\beta(\sigma, G\sigma) \ge 1$.*

Then G has a unique fixed point in Ω.

Proof. The proof follows from Theorem 3.2 by taking $H = G$. ☐

Example 3.4. *Let $\Omega = [0,\infty)$, and let $d : \Omega\times\Omega \to [0,\infty)$ be defined by*

$$d(\sigma,\delta) = \begin{cases} \max\{\sigma,\delta\} & \textit{if} \quad \sigma \ne \delta, \\ 0 & \textit{if} \quad \sigma = \delta. \end{cases}$$

We define $G, H : \Omega \to \Omega$ by $G(\sigma) = \frac{\rho}{4}$ and $H(\sigma) = \frac{\rho}{5}$ for all $\rho \in \Omega$. Let G and H are continuous self-mappings on Ω and $\alpha, \beta : \Omega\times\Omega \to [0,\infty)$ are two mappings defined by

$$\alpha(\sigma,\delta) = \begin{cases} 1 & \textit{if} \quad \sigma,\delta \in [0,1], \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$\beta(\sigma,\delta) = \begin{cases} 1 & \textit{if} \quad \sigma,\delta \in [0,1], \\ 0 & \textit{otherwise}. \end{cases}$$

We now define $\zeta : [0,\infty)\times[0,\infty) \to [0,\infty)$ by $\zeta(\delta,\sigma) = \frac{1}{2}\psi(\sigma) - \psi(\delta)$, for all $\sigma,\delta \in [0,\infty)$ and $\psi(\delta) = \frac{\delta}{4}$. Now

$$\begin{aligned} &\frac{1}{2}\min\{d(\sigma, G\sigma), d(\delta, H\delta)\} \le d(\sigma,\delta) \quad \textit{implies} \\ &\zeta(\alpha(\sigma, G\sigma)\beta(\delta, H\delta)d(G\sigma, H\delta), M(\sigma,\delta)) \\ &= \frac{1}{2}\psi(M(\sigma,\delta)) - \psi(\alpha(\sigma, G\sigma)\beta(\delta, H\delta)d(G\sigma, H\delta)) \\ &= \frac{1}{2}\psi(M(\sigma,\delta)) - \psi(d(G\sigma, H\delta)) \\ &< \frac{1}{8}M(\sigma,\delta) - \frac{1}{4}d(G\sigma, H\delta)) \ge 0, \end{aligned}$$

where $\zeta \in Z_\psi$ and

$$\begin{aligned} &M(\sigma,\delta) \\ &= \max\Big\{d(\sigma,\delta), d(\sigma, G\sigma), d(\delta, H\delta), \\ &\frac{d(\sigma, G\sigma)d(\sigma, H\delta) + d(\delta, H\delta)d(\delta, G\sigma)}{1 + d(\sigma, G\sigma) + d(\delta, H\delta)}, \\ &\frac{d(\sigma, G\sigma)d(\sigma, H\delta) + d(\delta, H\delta)d(\delta, G\sigma)}{1 + d(\sigma, H\delta) + d(\delta, G\sigma)}\Big\} \\ &+ L\min\{d(\sigma, G\sigma), d(\delta, H\delta), d(\sigma, H\delta), d(\delta, G\sigma)\}. \end{aligned}$$

Therefore, for $\sigma,\delta \in [0,1]$ and $L \ge 0$ the pair (G, H) is a Suzuki - type rational Z_ψ contraction. In either case $\alpha(\sigma,\delta) = 0$ and $\beta(\sigma,\delta) = 0$ then pair (G, H) is a Suzuki - type rational Z_ψ contraction.

As a result, the presumptions of Theorem 3.2 are all met, and G and H have a common fixed point in Ω.

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References:

[1] B. Samet, C. Vetro, P. Vetro, Fixed point Theorems for α-ψ- contractive type mappings, Nonlinear Anal., 75:2154-2165, 2012.

[2] E. Karapinar, B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal., 2012:17 pages, 2012.

[3] F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theory for simulation functions, Filomat, 29:1189-1194, 2015.

[4] H. Argoubi, B. Samet, C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, J. Nonlinear Sci. Appl., 8:1082-1094, 2015.

[5] Gh. Heidary J, A. Farajzadeh, M. Azhini, F. Khojasteh, A New Common Fixed Point Theorem for Suzuki Type Contractions via Generalized ψ-simulation Functions, Sahand Communications in Mathematical Analysis, 16(1):129-148, 2019.

[6] S. Radenović, S. Chandok. Simulation type functions and coincidence points. Filomat, 32(1):141-147, 2018.

[7] D. Kitkuan, P. Bunpatcharacharoen, Coincidence point theorems for multi-valued mapping b-metric spaces via digraphs. Advances in Mathematics: Scientific Journal, 10(6):2785-2797, 2021.

[8] A. Padcharoen, J.K. Kim, Berinde type results via simulation functions in metric spaces. Nonlinear Functional Analysis and Applications, 25(3): 511-523, 2020.

[9] S. Chandok, A. Chanda, L.K. Dey, M. Pavlović, S. Radenović, Simulation Functions and Geraghty Type Results, Bol. Soc. Paran. Mat. 39(1):35-50, 2021.

[10] A. Padcharoen, P. Kumam, P. Saipara, P. Chaipunya, Generalized Suzuki type $\mathcal{Z}$-contraction in complete metric spaces, Kragujevac J. Math. 42(3):419-430, 2018.

[11] D. Kitkuan, S. Saelee, Generalized contractions via $\mathcal{Z}$-contraction, Nonlinear Functional Analysis and Applications, 27(3):587-601, 2022.

[12] X. Liu, A.H. Ansari, S. Chandok, S. Radenović, On some results in metric spaces using auxiliary simulation functions via new functions, J. Comput. Anal. Appl., 24:1103-1114, 2018.

[13] M.A. Kutbi, W. Sintunavarat, On new fixed point results for (α, ψ, ζ)-contractive multivalued mappings on α-complete metric spaces and their consequences, Fixed Point Theo. and Appl. 2015.

[14] N. Hussain, M.A. Kutbi, P. Salimi, Fixed point Theory in α-complete metric spaces with applications, Abstract and Applied Analysis 2014, Article ID. 280817.

[15] A. Padcharoen, P. Kumam, D. Gopal, Coincidence and periodic point results in a modular metric space endowed with a graph and applications, Creative Mathematics and Informatics, 26(1):95-104, 2017.

[16] S. Radenović, Z. Kadelburg, D. Jandrlixex, A. Jandrlixex, Some results on weak contraction maps, Bull. Iran. Math. Soc., 38:625-645, 2012.