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Banach's C^*-algebra contraction mappings for α-fuzzy fixed point theorems in C^*-algebra-valued b-metric spaces

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Abstract

In this paper, we introduce the new concept of multivalued contraction mappings extend to C^*-algebra-valued b-metric spaces. The existence of α-fuzzy fixed points for nonlinear mappings in C^*-algebra-valued b-metric spaces. Then, α-fuzzy fixed point results for Banach's C^*-algebra contraction mappings are obtained. Also, we give some examples for support our main results.

Keywords: C^*-algebra-valued b-metric spaces, α-fuzzy fixed point, Banach's C^*-algebra contraction mappings.
2010 MSC: 47H09, 47H10, 47H30.

1. Introduction

In 1981, Heilpern [5] studied and introduced the notion of fuzzy contraction mappings and proved a fixed point theorems. Frigon and Regan [4] generalized the Heilpern's theorem under a contractive condition of Nadler's fixed point theorem for 1-level sets (i.e., $[Tx]_1$) of a fuzzy mapping T, where 1-level sets are not a convex set and a compact set. Amemiya and Takahashi [2] studied some properties of contraction type set-valued and others have studied fuzzy contraction mappings to obtain fixed points of fuzzy contraction

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Let $F : X \to C(X)$ be given by $Fx = [Tx]_1$. Notice that F satisfies the conditions of Nadler's fixed point theorem. As a result there exists $x \in X$ with $x \in Fx = [Tx]_1$.

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mappings to see [5, 11]. Phiangsungnoen [9] studied fuzzy fixed point theorems for multivalued fuzzy contractions in b-metric spaces.

Recently, Ma *et al.* [7] introduced the notion of C^*-algebra-valued metric space and presented the Banach contraction principle for C^*-algebra-valued metric space. Lather, Kamran [6] generalized the notion of C^*-algebra-valued b-metric space and proved certain fixed point results.

In this paper, we using the concept of C^*-algebra-valued b-metric space and consequently the existence and an α-fuzzy fixed point for such a mapping are obtained. Our results substantially generalize several comparable results from the literature corresponding to see [8, 9]. In the finall, we give some examples for support our main theorems.

2. Preliminaries

2.1. C^*-algebra-valued b-metric spaces

We shall study the basic definitions and properties about C^*-algebra-valued b-metric spaces.

Definition 2.1 ([6]). Let $\mathbb{A}$ be a C^*-algebra, and X be an arbitrary nonempty set, $b \in \mathbb{A}$ be shch that $\| b \| \geq 1$. A mapping $d_b : X^2 \to \mathbb{A}_+$ is called a C^*-algebra-valued b-metric on X if for all $x, y, z \in \mathbb{A}$, the following properties are satisfied:

(i) $d_b(x, y) = 0_A$ if and only if $x = y$;
(ii) $d_b(x, y) = d_b(y, x)$;
(iii) $d_b(x, z) \preceq s[d_b(x, y) + d_b(y, z)]$.

The $3-tuple$ $(X, \mathbb{A}, d_b)$ is called a C^*-algebra-valued b-metric space with coefficient s.

Example 2.2 ([6]). Let $X = l_p$ be the set of sequences $\{x_n\}$ in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ and $0 < p < 1$. Let $\mathbb{A} = M_2(\mathbb{R})$. For $x = x_n, y = y_n \in l_p$, define $d_b : X^2 \to \mathbb{A}_+$ as follows:

$$d_b(x, y) = \begin{pmatrix} (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} & 0 \\ 0 & (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} \end{pmatrix}.$$

Then, $(X, \mathbb{A}, d_b)$ is a C^*-algebra-valued b-metric space with coefficient $s = \begin{pmatrix} 2^{\frac{1}{p}} & 0 \\ 0 & 2^{\frac{1}{p}} \end{pmatrix}$ such that $\| s \| = 2^{\frac{1}{p}}$.

Thus

$$(\sum_{n=1}^{\infty} |x_n - z_n|^p)^{\frac{1}{p}} \leq (2^p)^{\frac{1}{p}} \Big[(\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} + (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} \Big].$$

Definition 2.3 ([6]). Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ with respect to the algebra $\mathbb{A}$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\| d_b(x_n, x) \| < \varepsilon$ for every $n > N$. Symbolically, we then write $\lim_{n\to\infty} x_n = x$.

Definition 2.4 ([6]). A sequence $\{x_n\}$ in X is called a Cauchy sequence with respect to $\mathbb{A}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\| d_b(x_n, x_m) \| < \varepsilon$ for every n, $m > N$.

Definition 2.5 ([6]). A C^*-algebra-valued b-metric space $(X, \mathbb{A}, d_b)$ is complete if every Cauchy sequence in X is convergent with respect to $\mathbb{A}$.

Definition 2.6 ([6]). Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. A contraction on X is a mapping $T : X \to X$ if there exists an $a \in \mathbb{A}$ with $\| a \| < 1$

$$d_b(Tx, Ty) \preceq a^* d_b(x, y) a \tag{2.1}$$

for all $x, y \in X$.

2.2. An α-fuzzy fixed points

Let X be a nonempty set and $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. A fuzzy set M in X is characterized by a membership function $M : X \to [0,1]$ such that each element $x \in X$ is associated with a real number $M(x) \in [0,1]$. Let $F(X)$ be a collection of all fuzzy subsets of X and M be a fuzzy set in X. If $\alpha \in (0,1]$, then the α-level set M_α of M is defined as $M_\alpha = \{x \in X : M(x) \geq \alpha\}$. For $\alpha = 0$, we have $M_0 = \overline{\{x \in X : M(x) > 0\}}$, where $\overline{M}$ denotes the closure of a set M in (X, d). A fuzzy set$M(x) \leq N(x)$ if and only if M is said to be more accurate than fuzzy set N, define by $M \to N$ for each x in X. It is obvious that if $0 < \alpha \leq \beta \leq 1$, then $M_\beta \subseteq M_\alpha$.

Now, for $x \in X$, $M, N \in F(X)$, $\alpha \in [0,1]$ and $[M]_\alpha, [N]_\alpha \in CB(X)$. Define

$$W_\alpha(X) = \{M \in I^X : M_\alpha \text{ is nonempty and compact}\}.$$

For $M, N \in W_\alpha(X)$ and $\alpha \in [0,1]$, let

$$\begin{aligned} p_\alpha(M,N) &= \inf\{d(x,y); x \in M_\alpha; y \in N_\alpha\}, \\ H(M,N) &= \max\{\sup_{x \in M_\alpha} d(x, N_\alpha), \sup_{y \in N_\alpha} d(y, M_\alpha)\}, \\ d_\infty(M,N) &= \sup_\alpha H(M,N). \end{aligned}$$

Note that p_α is a nondecreasing mapping of α and H a metric on $W_\alpha(X)$.

We write $p(x, N)$ instead of $p(\{x\}, N)$. A fuzzy set M in a metric linear space V is said to be an approximate quantity if and only if $[M]_\alpha$ is compact and convex in V for each $\alpha \in [0,1]$ and $\sup_{x \in V} M(x) = 1$. We denote the collection of all approximate quantities in a metric linear space X by $W_\alpha(X)$.

We following the concept of a α-fuzzy fixed point:

Definition 2.7. Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space and T be a fuzzy mappings from X into $F(X)$. A point z in X is said to be an *α-fuzzy fixed point* of T if $z \in [Tz]_\alpha$.

3. Main results

From the concept of C^*-algebra-valued contractions in [7], we defined the new notion of *Banach's C^*-algebra contractions* in C^*-algebra-valued b-metric space as follows.

Definition 3.1. Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. A mapping $T : X \to F(X)$ is said to be an *Banach's C^*-algebra contraction* on $(X, \mathbb{A}, d_b)$ if there exists a $a \in \mathbb{A}$ with $\| a \| < 1$

$$H\left([Tx]_\alpha, [Ty]_\alpha\right) \preceq a^* d_b(x,y) a \quad \text{for all} \quad x, y \in X. \tag{3.1}$$

Now, we start and prove some α-fuzzy fixed point results for Banach's C^*-algebra contractions as follows.

Theorem 3.2. *Let X be a nonempty set and let $(X, \mathbb{A}, d_b)$ is a complete C^*-algebra-valued b-metric space with coefficient s. A fuzzy mapping $T : X \to F(X)$ be a Banach's C^*-algebra contraction, there exists $a \in \mathbb{A}$ such that $\| s \| \| a \|^2 < 1$. If there exists a point $x_\alpha \in X$ such that $x_\alpha \subset [Tx]_\alpha$, then T has an α-fuzzy fixed point.*

Proof. Let x_0 be arbitrary point in X. We define the sequence in X as $x_{n+1} \in [Tx_n]_\alpha$ for any $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0} \in [Tx_{n_0}]_\alpha$. This proves that x_{n_0} is an α-fuzzy fixed point of T.

Assume that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N} \cup \{0\}$. From (3.1) that for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} 0_A \preceq d_b(x_n, x_{n+1}) &\preceq H([Tx_{n-1}]_\alpha, [Tx_n]_\alpha) \\ &\preceq a^* d_b(x_{n-1}, x_n) a \\ &= a^* d_b([Tx_{n-2}]_\alpha, [Tx_{n-1}]_\alpha) a \end{aligned}$$

$$\begin{aligned}
&\preceq (a^*)^2 d_b(x_{n-2}, x_{n-1})a^2\\
&= (a^*)^2 d_b([Tx_{n-3}]_\alpha, [Tx_{n-2}]_\alpha)a^2\\
&\preceq (a^*)^3 d_b(x_{n-3}, x_{n-2})a^3\\
&\vdots\\
&\preceq (a^*)^n d_b(x_0, x_1)a^n\\
&= (a^*)^n M a^n, \text{ where } M = d_b(x_0, x_1).
\end{aligned}$$

For $m, n \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned}
d_b(x_m, x_n) &\preceq s d_b(x_m, x_{m+1}) + s^2 d_b(x_{m+1}, x_{m+2}) + \ldots + s^{n-m-1} d_b(x_{n-2}, x_{n-1})\\
&+ s^{n-m} d_b(x_{n-1}, x_n)\\
&\preceq s(a^*)^m M a^m + s^2 (a^*)^{m+1} M a^{m+1} + \ldots + s^{n-m-1}(a^*)^{n-2} M a^{n-2}\\
&+ s^{n-m}(a^*)^{n-1} M a^{n-1}\\
&= s[(a^*)^m M a^m + s(a^*)^{m+1} M a^{m+1} + \ldots + s^{n-m-2}(a^*)^{n-2} M a^{n-2}\\
&+ s^{n-m-1}(a^*)^{n-1} M a^{n-1}]\\
&= s\sum_{i=m}^{n-1} s^{i-m}(a^*)^i M a^i\\
&= s\sum_{i=m}^{n-1} s^{i-m}(a^*)^i M^{\frac{1}{2}} M^{\frac{1}{2}} a^i\\
&= s\sum_{i=m}^{n-1} s^{i-m}(M^{\frac{1}{2}} a^i)^*(M^{\frac{1}{2}} a^i)\\
&= s\sum_{i=m}^{n-1} s^{i-m}|M^{\frac{1}{2}} a^i|^2\\
&\preceq \Big|\Big|s\sum_{i=m}^{n-1} s^{i-m}|M^{\frac{1}{2}} a^i|^2\Big|\Big| I\\
&\preceq ||s||\sum_{i=m}^{n-1} ||s^{i-m}||||M^{\frac{1}{2}}||^2||a^i||^2 I\\
&\preceq ||s||^{1-m}||M^{\frac{1}{2}}||^2 \sum_{i=m}^{n-1} ||s^i||||a^i||^2 I\\
&\preceq ||s||^{1-m}||M^{\frac{1}{2}}||^2 \sum_{i=m}^{n-1} ||s^i||||a^2||^i I\\
&\preceq ||s||^{1-m}||M^{\frac{1}{2}}||^2 \sum_{i=m}^{n-1} \Big(||s||||a^2||\Big)^i I \longrightarrow 0_A \ (m \to \infty).
\end{aligned}$$

Since, $\| s \|\| a \|^2 < 1$ implies that $\Big(\| s \|\| a \|^2 \Big)^m \to 0$, then the series $\sum_{i=m}^{n-1} \Big(\| s \|\| a \|^2 \Big)^i$ is a converges, we get that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X with respect to $\mathbb{A}$. Since $(X, \mathbb{A}, d_b)$ is complete, there exists

$z \in X$, we have $\lim_{n\to+\infty} x_n = z$. Next, we will show that z is an α-fuzzy fixed point. We claim that $z \in [Tz]_\alpha$, by condition (iii) the triangle inequality, we have

$$\begin{aligned} d_b(z,[Tz]_\alpha) &\preceq s[d_b(z,x_{n+1}) + d_b(x_{n+1},[Tz]_\alpha)] \\ &\preceq s[d_b(z,x_{n+1}) + H([Tx_n]_\alpha,[Tz]_\alpha)] \\ &\preceq s[d_b(z,x_{n+1}) + a^* d_b(x_n,z)a] \longrightarrow 0_A, \quad (n\to\infty). \end{aligned}$$

Therefore, we get $d_b(z,[Tz]_\alpha) = 0_A$ and $[Tz]_\alpha$ is closed, that is, $z \in [Tz]_\alpha$. Thus z is an α-fuzzy fixed point of T. This completes the proof. □

By taking $\mathbb{A} = \mathbb{R}$, in Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$, let $T : X \to F(X)$, $\alpha : X \to (0,1]$ such that $[Tx]_\alpha$ is a nonempty closed subsets of X, for all $x \in X$ such that*

$$H([Tx]_\alpha,[Ty]_\alpha) \leq kd(x,y),$$

for all $x,y \in X$, where $0 < k < 1$. Assume that $k < \frac{1}{s}$, then T has an α-fuzzy fixed point.

Corollary 3.4. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$, let $T : X \to F(X)$, $\alpha : X \to (0,1]$ such that $[Tx]_\alpha$ is a nonempty closed subsets of X, for all $x \in X$ and $\psi \in \Psi_b$, such that*

$$H([Tx]_\alpha,[Ty]_\alpha) \leq \psi(d(x,y)),$$

for all $x,y \in X$. Assume that $k < \frac{1}{s}$, then T has an α-fuzzy fixed point.

Next, we give some examples to support the validity of Theorem 3.2.

Example 3.5. Let $X = [0,3]$ and define a C^*-algebra-valued b-metric d_b on $X^2 \to \mathbb{A}$ by $d_b(x,y) = |x-y|^2 I$ for all $x,y \in X$. Then $(X,\mathbb{A},d_b)$ is a complete C^*-algebra-valued b-metric space with coefficient $s = 2$, but $(X,\mathbb{A},d_b)$ is not a C^*-algebra-valued metric space. Define a fuzzy mapping $T : X \to F(X)$ by

$$(Tx)(t) = \begin{cases} 0, & \text{if } t \in [0,\frac{3}{4}), \\ \dfrac{3}{4}, & \text{if } t \in [\frac{3}{4},2], \\ \dfrac{1}{t+1}, & \text{if } t \in (2,3]. \end{cases}$$

Let $\alpha = \frac{3}{4}$, now we get

$$[Tx]_{\frac{3}{4}} = [\frac{3}{4},1].$$

Note that, for $x,y \in X$, we have

$$H([Tx]_{\frac{3}{4}},[Ty]_{\frac{3}{4}}) = \frac{9}{16}|x-y|^2 I = \frac{3}{4}|x-y|^2 I\frac{3}{4} = \frac{3}{4}d_b(x,y)\frac{3}{4},$$

where $\| a \| = \frac{3}{4} \in (0,1)$.

Thus, all the conditions of Theorem 3.2 are satisfied. We have $x = \frac{3}{4} \in X := [0,2]$ is an α-fuzzy fixed point of T. Indeed, for $x = \frac{3}{4}$, we have $x_\alpha \subset [Tx]_\alpha$ as $(T\frac{3}{4})(\frac{3}{4}) \geq \frac{3}{4}$, that is, $\frac{3}{4} \in [T\frac{3}{4}]_{\frac{3}{4}}$. Thus $x = \frac{3}{4}$ is a $(\frac{3}{4})$-fuzzy fixed point of T.

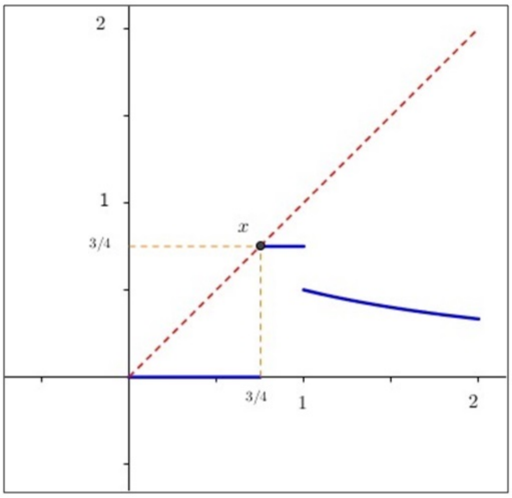

Figure 1: $(Tx)(t)$

4. Competing interests

The author declares that he has no competing interest.

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