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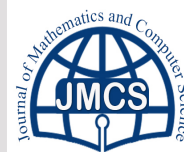
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Strong convergence of modified viscosity implicit approximation methods for asymptotically nonexpansive mappings in complete CAT(0) spaces

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Abstract

In this paper, we introduce a modified viscosity implicit iteration for asymptotically nonexpansive mappings in complete CAT(0) spaces. Under suitable conditions, we prove some strong convergence to a fixed point of an asymptotically nonexpansive mapping in a such space which is also the solution of variational inequality. Moreover, we illustrate some numerical example of our main results. Our results extend and improve some recent result of Yao et al. [Y.-H. Yao, N. Shahzad, Y.-C. Liou, Fixed Point Theory Appl., **2015** (2015), 15 pages] and Xu et al. [H.-K. Xu, M. A. Alghamdi, N. Shahzad, Fixed Point Theory Appl., **2015** (2015), 12 pages]. ©2017 All rights reserved.

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1. Introduction

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that

$$c(0) = x, \quad c(r) = y, \quad d(c(t), c(s)) = |t - s|$$

for all $s, t \in [0, r]$. In particular, c is an isometry and $d(x, y) = r$. The image of c is called a *geodesic (or metric) segment* joining x and y . When it is unique, this geodesic is denoted by $[x, y]$. We denote the point $w \in [x, y]$ such that $d(x, w) = \alpha d(x, y)$ by $w = (1 - \alpha)x \oplus \alpha y$, where $\alpha \in [0, 1]$.

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The space (X, d) is called a *geodesic space* if any two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $D \subseteq X$ is said to be *convex* if D includes geodesic segment joining every two points of itself. A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A *comparison triangle* for geodesic triangle (or $\Delta(x_1, x_2, x_3)$) in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane \mathbb{R}^2 such that

$$d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$$

for $i, j \in \{1, 2, 3\}$. A geodesic metric space is called a CAT(0) space ([5]) if all geodesic triangles satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(0) inequality if, for all $x, y \in \Delta$ and all comparison points,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, which is denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{1.1}$$

The inequality (1.1) is called the (CN) *inequality* (for more details, see Bruhat and Titz [6]). In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well-known that all complete, simply combined Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings, Pre-Hilbert spaces, R-trees ([5]), the complex Hilbert ball with a hyperbolic metric is a CAT(0) space. Further, Complete CAT(0) spaces are called *Hadamard* spaces.

Let C be a nonempty subset of a complete CAT(0) space X . Then a mapping $T : C \rightarrow C$ is called

- (1) nonexpansive if and only if $d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C;$
- (2) asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n d(x, y), \quad \forall n \geq 1, \quad x, y \in C.$

A point $x \in X$ is called a fixed point of T if $x = Tx$. We will denote by $F(T)$ the set of fixed points of T . Kirk [12] proved the existence theorem of fixed points for asymptotically nonexpansive mappings in CAT(0) spaces.

A mapping $f : C \rightarrow C$ is called a contraction with coefficient $k \in [0, 1)$ if and only if

$$d(f(x), f(y)) \leq kd(x, y), \quad \forall x, y \in C.$$

f has a unique fixed point when C is a nonempty, closed, and subset of a complete metric space was guaranteed by Banach’s contraction principle [3]. The existence theorems of fixed points and convergence theorems for various mappings in CAT(0) spaces have been investigated by many authors [7–10, 13]. Obviously every contraction mapping is nonexpansive and every nonexpansive mapping is asymptotically nonexpansive with the sequence $\{k_n = 1\}$ for all $n \geq 1$.

One of the powerful numerical methods is implicit midpoint method for solving ordinary differential equations and differential algebraic equations. For a study discussion related to this numerical method we refer to [1, 2, 14, 15].

Very recently, Xu et al. [18] introduced the following viscosity implicit midpoint method for a nonexpansive mapping $T : H \rightarrow H$ in a Hilbert space H :

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right)$$

for all $n \geq 0$, where $\alpha_n \in (0, 1)$ and $f : H \rightarrow H$ is a contraction. Under suitable conditions, they proved that the sequence $\{x_n\}$ converges strongly to a fixed point of a nonexpansive mapping, which is also a unique solution of the following variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T).$$

Motivated and inspired by Xu et al. [18], we study and introduce the following modified viscosity implicit iteration for an asymptotically nonexpansive mapping in CAT(0) space X :

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) T^n (\beta_n x_n \oplus (1 - \beta_n) x_{n+1}) \tag{1.2}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$, $T^n : X \rightarrow X$ be an asymptotically nonexpansive mapping and $f : X \rightarrow X$ is a contraction.

The purpose of this paper, we prove strong convergence theorems of the modified viscosity implicit iteration process for an asymptotically nonexpansive mapping in CAT(0) space under suitable conditions. We also show that the limit of the sequence $\{x_n\}$ generated by (1.2) solves the solution of the following variational inequality

$$\langle \overrightarrow{pf(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geq 0, \quad \forall p \in F(T).$$

Furthermore, we illustrate some numerical examples for support our main results.

2. Preliminaries

In this section, we always suppose that X is a CAT(0) space and write $(1 - t)x \oplus ty$ for the unique point w in the geodesic segment joining from x to y which is, $[x, y] = \{(1 - \lambda)x \oplus \lambda y : \lambda \in [0, 1]\}$,

$$d(w, x) = \lambda d(x, y), \quad \text{and} \quad d(w, y) = (1 - \lambda)d(x, y).$$

Lemma 2.1 ([11]). *Let K be a CAT(0) space. For all $x, y, z \in X$ and $\lambda, \gamma \in [0, 1]$, we have the followings:*

- (i) $d(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)$;
- (ii) $d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y)$;
- (iii) $d(\lambda x \oplus (1 - \lambda)y, \gamma x \oplus (1 - \gamma)y) = |\lambda - \gamma|d(x, y)$;
- (iv) $d(\lambda x \oplus (1 - \lambda)y, tu \oplus (1 - \lambda)w) \leq \lambda d(x, u) + (1 - \lambda)d(y, w)$.

Lemma 2.2 ([17]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \xi_n$$

for all $n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\xi_n\}$ is a sequence in \mathbb{R} such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \frac{\xi_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\xi_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

The concept of quasilinearization in X was introduced by Berg and Nikolaev [4].

Denote a pair $(p, q) \in X \times X$ by \overrightarrow{pq} and call it a vector. Then quasi-linearization is defined as a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R}$ such that

$$\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle = \frac{1}{2}(d^2(p, s) + d^2(q, r) - d^2(p, r) - d^2(q, s))$$

for all $p, q, r, s \in X$.

Let C be a nonempty closed convex subset of a complete CAT(0) space X . The metric projection $P_C : X \rightarrow C$ is defined by

$$w = P_C(x) \iff d(w, x) = \inf\{d(y, x) : y \in C\}$$

for all $x \in X$.

Lemma 2.3 ([4]). *Let C be a nonempty closed and convex subset of a complete CAT(0) space X , $x \in X$ and $w \in C$. Then $w = P_C(x)$ if and only if*

$$\langle \overrightarrow{yw}, \overrightarrow{wx} \rangle \geq 0$$

for all $y \in C$.

Lemma 2.4 ([16]). *Let X be a CAT(0) space, C be a nonempty closed and convex subset of X , and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n \rightarrow \infty} k_n = 1$. For any contraction $f : C \rightarrow C$ and $\lambda \in (0, 1)$, let $x_\lambda \in C$ be the unique fixed point of the contraction $x \mapsto \lambda f(x) \oplus (1 - \lambda)Tx$, i.e.,*

$$x_\lambda = \lambda f(x_\lambda) \oplus (1 - \lambda)Tx_\lambda.$$

Then $\{x_n\}$ converges strongly as $\lambda \rightarrow 0$ to a point \tilde{p} such that

$$\tilde{p} = P_{F(T)}f(\tilde{p}),$$

which is the unique solution to the following variational inequality:

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{\tilde{p}\tilde{p}} \rangle \geq 0$$

for all $p \in F(T)$.

3. Main results

Now, we prove the main results as follows.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n \rightarrow \infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitrary initial point $x_1 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T^n(\beta_n x_n \oplus (1 - \beta_n)x_{n+1}) \tag{3.1}$$

for all $n \geq 0$, where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in interval $(0, 1)$, satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) $\lim_{n \rightarrow \infty} \frac{k_n - 1}{\alpha_n} = 0$;
- (iv) $\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} \rightarrow 0$, as $n \rightarrow \infty$;
- (v) T satisfies the asymptotically regular $\lim_{n \rightarrow \infty} d(x_n, T^n x_n) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\tilde{p} = P_{F(T)}f(\tilde{p})$, which is a fixed point of T and, also, it is a solution of the following variational inequality:

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{\tilde{p}\tilde{p}} \rangle \geq 0, \quad \forall p \in F(T). \tag{3.2}$$

Proof. We divide the proof into three steps.

Step 1. First, we prove that for all $v \in C$, the mapping defined by

$$x \rightarrow T_v(x) := \alpha f(v) \oplus (1 - \alpha)T^n(\beta v \oplus (1 - \beta)x)$$

for all $x \in C$, $\alpha, \beta \in (0, 1)$ and f is a contraction mapping with the contractive constant $(1 - \alpha)k_n(1 - \beta)$. Indeed, it follows from Lemma 2.1 that, for all $x, y \in C$

$$d(T_v x, T_v y) = d(\alpha f(v) \oplus (1 - \alpha)T^n(\beta v \oplus (1 - \beta)x), \alpha f(v) \oplus (1 - \alpha)T^n(\beta v \oplus (1 - \beta)y))$$

$$\begin{aligned} &\leq \alpha d(f(v), f(v)) + (1 - \alpha)d(T^n(\beta v \oplus (1 - \beta)x), T^n(\beta v \oplus (1 - \beta)y)) \\ &\leq (1 - \alpha)k_n(\beta d(v, v) + (1 - \beta)d(x, y)) \\ &\leq (1 - \alpha)k_n(1 - \beta)d(x, y). \end{aligned}$$

It follows that $0 < (1 - \alpha)k_n(1 - \beta) < 1$. This implies that the mapping $T_v : C \rightarrow C$ is contraction with a constant $(1 - \alpha)k_n(1 - \beta)$. Thus the sequence $\{x_n\}$ defined by (3.1) is well-defined.

Step 2. Next, we prove that the sequence $\{x_n\}$ is bounded. By condition (iii), for any $0 < \varepsilon < 1 - \alpha$ sufficient large $n \geq 0$, we have $k_n - 1 \leq \alpha_n \varepsilon$. For all $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1 - \alpha_n)T^n(\beta_n x_n \oplus (1 - \beta_n)x_{n+1}), p) \\ &\leq \alpha_n d(f(x_n), p) + (1 - \alpha_n)d(T^n(\beta_n x_n \oplus (1 - \beta_n)x_{n+1}), p) \\ &\leq \alpha_n(d(f(x_n), f(p)) + d(f(p), p)) + (1 - \alpha_n)k_n(\beta_n d(x_n, p) + (1 - \beta_n)d(x_{n+1}, p)) \\ &\leq \alpha_n k d(x_n, p) + \alpha_n d(f(p), p) + k_n(1 - \alpha_n)\beta_n d(x_n, p) + k_n(1 - \alpha_n)(1 - \beta_n)d(x_{n+1}, p), \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1}, p) &\leq \frac{\alpha_n k + \beta_n k_n - \alpha_n \beta_n k_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n} d(x_n, p) + \frac{\alpha_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n} d(f(p), p) \\ &= \left(1 - \frac{(1 - k_n + \alpha_n k_n - \alpha_n k)}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n}\right) d(x_n, p) + \frac{\alpha_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n} d(f(p), p) \\ &= \left(1 - \frac{-(k_n - 1 - \alpha_n k_n + \alpha_n k)}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n}\right) d(x_n, p) + \frac{\alpha_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n} d(f(p), p) \\ &\leq \left(1 - \frac{-(\alpha_n \varepsilon - \alpha_n k_n + \alpha_n k)}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n}\right) d(x_n, p) + \frac{\alpha_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n} d(f(p), p) \\ &= \left(1 - \frac{(k_n - k - \varepsilon)\alpha_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n}\right) d(x_n, p) \\ &\quad + \frac{(k_n - k - \varepsilon)\alpha_n}{(1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n)k_n)(k_n - k - \varepsilon)} d(f(p), p) \\ &\leq \left(1 - \frac{(1 - k - \varepsilon)\alpha_n}{\alpha_n + \beta_n - \alpha_n \beta_n}\right) d(x_n, p) + \frac{(1 - k - \varepsilon)\alpha_n}{(\alpha_n + \beta_n - \alpha_n \beta_n)(1 - k - \varepsilon)} d(f(p), p) \\ &\leq \max\left\{d(x_n, p), \frac{d(f(p), p)}{(1 - k - \varepsilon)}\right\}. \end{aligned}$$

By mathematical induction, we can prove that

$$d(x_n, p) \leq \max\left\{d(x_0, p), \frac{d(f(p), p)}{(1 - k - \varepsilon)}\right\}$$

for all $n \geq 0$. This implies that the sequence $\{x_n\}$ is bounded, so $\{f(x_n)\}$ and $\{T^n(\beta_n x_n \oplus (1 - \beta_n)x_{n+1})\}$ are also bounded.

Step 3. Next, we prove that the sequence $\{x_n\}$ converges strongly to $\tilde{p} = P_{F(T)}f(\tilde{p})$. Set

$$w_n = \alpha_n f(w_n) \oplus (1 - \alpha_n)T^n(w_n)$$

for all $n \geq 0$. By Lemma 2.4, the sequence $\{w_n\}$ converges strongly as $n \rightarrow \infty$ to a point $\tilde{p} = P_{F(T)}f(\tilde{p})$, which is the unique solution to the variational inequality (3.2).

On the other hand, it follows from (3.1) and Lemma 2.1, that

$$\begin{aligned} d(x_{n+1}, w_n) &= d(\alpha_n f(x_n) \oplus (1 - \alpha_n)T^n(\beta_n x_n \oplus (1 - \beta_n)x_{n+1}), \alpha_n f(w_n) \oplus (1 - \alpha_n)T^n w_n) \\ &\leq \alpha_n d(f(x_n), f(w_n)) + (1 - \alpha_n)d(T^n(\beta_n x_n \oplus (1 - \beta_n)x_{n+1}), T^n w_n) \\ &\leq \alpha_n k d(x_n, w_n) + (1 - \alpha_n)k_n(\beta_n d(x_n, w_n) + (1 - \beta_n)d(x_{n+1}, w_n)), \end{aligned}$$

which implies that

$$\begin{aligned}
 d(x_{n+1}, w_n) &\leq \frac{\alpha_n k + \beta_n k_n - \alpha_n \beta_n k_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n) k_n} d(x_n, w_n) \\
 &= \left(1 - \frac{-(k_n - 1 - \alpha_n k_n + \alpha_n k)}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n) k_n}\right) d(x_n, w_n) \\
 &\leq \left(1 - \frac{-(\alpha_n \varepsilon - \alpha_n k_n + \alpha_n k)}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n) k_n}\right) d(x_n, w_n) \\
 &= \left(1 - \frac{(k_n - k - \varepsilon) \alpha_n}{1 - (1 - \alpha_n - \beta_n + \alpha_n \beta_n) k_n}\right) d(x_n, w_n) \\
 &\leq \left(1 - \frac{(1 - k - \varepsilon) \alpha_n}{\alpha_n + \beta_n - \alpha_n \beta_n}\right) d(x_n, w_n) \\
 &\leq (1 - (1 - k - \varepsilon) \alpha_n) (d(x_n, w_{n-1}) + d(w_{n-1}, w_n)) \\
 &\leq (1 - (1 - k - \varepsilon) \alpha_n) d(x_n, w_{n-1}) + d(w_{n-1}, w_n).
 \end{aligned} \tag{3.3}$$

In order to use Lemma 2.2, it should be proved that

$$\limsup_{n \rightarrow \infty} \frac{d(w_{n-1}, w_n)}{(1 - k - \varepsilon) \alpha_n} \leq 0. \tag{3.4}$$

In fact, by Lemma 2.1, we have

$$\begin{aligned}
 d(w_n, w_{n-1}) &= d(\alpha_n f(w_n) \oplus (1 - \alpha_n) T^n w_n, \alpha_{n-1} f(w_{n-1}) \oplus (1 - \alpha_{n-1}) T^n w_{n-1}) \\
 &\leq d(\alpha_n f(w_n) \oplus (1 - \alpha_n) T^n w_n, \alpha_n f(w_n) \oplus (1 - \alpha_n) T^n w_{n-1}) \\
 &\quad + d(\alpha_n f(w_n) \oplus (1 - \alpha_n) T^n w_{n-1}, \alpha_n f(w_{n-1}) \oplus (1 - \alpha_n) T^n w_{n-1}) \\
 &\quad + d(\alpha_n f(w_{n-1}) \oplus (1 - \alpha_n) T^n w_{n-1}, \alpha_{n-1} f(w_{n-1}) \oplus (1 - \alpha_{n-1}) T^n w_{n-1}) \\
 &\leq (1 - \alpha_n) d(T^n w_n, T^n w_{n-1}) + \alpha_n d(f(w_n), f(w_{n-1})) + |\alpha_n - \alpha_{n-1}| d(f(w_{n-1}), T^n w_{n-1}) \\
 &\leq (1 - \alpha_n) k_n d(w_n, w_{n-1}) + \alpha_n k d(w_n, w_{n-1}) + |\alpha_n - \alpha_{n-1}| M^*,
 \end{aligned}$$

where $M^* = \sup_{n \geq 1} d(f(w_{n-1}), T^n w_{n-1})$, which implies that

$$\begin{aligned}
 d(w_n, w_{n-1}) &\leq \frac{1}{1 - k_n + \alpha_n k_n - \alpha_n k} |\alpha_n - \alpha_{n-1}| M^* \\
 &= \frac{1}{-(k_n - 1 - \alpha_n k_n + \alpha_n k)} |\alpha_n - \alpha_{n-1}| M^* \\
 &\leq \frac{1}{-(\varepsilon - k_n + k) \alpha_n} |\alpha_n - \alpha_{n-1}| M^* \\
 &= \frac{1}{(k_n - k - \varepsilon) \alpha_n} |\alpha_n - \alpha_{n-1}| M^* \\
 &\leq \frac{1}{(1 - k - \varepsilon) \alpha_n} |\alpha_n - \alpha_{n-1}| M^*.
 \end{aligned}$$

By the condition (iv), we have

$$\limsup_{n \rightarrow \infty} \frac{d(w_{n-1}, w_n)}{(1 - k - \varepsilon) \alpha_n} \leq \limsup_{n \rightarrow \infty} \frac{|\alpha_n - \alpha_{n-1}|}{(1 - k - \varepsilon)^2 \alpha_n^2} M^* = 0.$$

Thus (3.4) is proved. By Lemma 2.2 and (3.3), we obtain

$$d(x_{n+1}, w_n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Since $w_n \rightarrow \tilde{p} = P_{F(T)} f(\tilde{p})$ and \tilde{p} is the unique solution of the variational inequality (3.2), we have $x_n \rightarrow \tilde{p}$ as $n \rightarrow \infty$ and \tilde{p} is also the unique solution of variational inequality (3.2). This completes the proof. \square

Remark 3.2. Since every Hilbert space is a complete CAT(0) space, Theorem 3.1 generalizes and improves the main results in Yao et al. [19] and Xu et al. [18].

Corollary 3.3. *Let C be nonempty closed and convex subset of a real Hilbert space H and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n \rightarrow \infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T^n(\beta_n x_n + (1 - \beta_n)x_{n+1})$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$ satisfy the conditions (i)-(v) as in Theorem 3.1. Then the sequence $\{x_n\}$ converges strongly to \tilde{p} such that

$$\tilde{p} = P_{F(T)} f(\tilde{p}),$$

which is the unique solution of the following variational inequality

$$\langle \tilde{p} - f(\tilde{p}), p - \tilde{p} \rangle \geq 0, \quad \forall p \in F(T). \tag{3.5}$$

From Theorem 3.1, if sequence $\{k_n := 1\}$, then $T : C \rightarrow C$ in Theorem 3.1 is a nonexpansive mapping, we can obtain the following results immediately.

Corollary 3.4. *Let C be nonempty closed and convex subset of a complete CAT(0) space X and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) T(\beta_n x_n \oplus (1 - \beta_n)x_{n+1})$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$ satisfy the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) $\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} \rightarrow 0$ as $n \rightarrow \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\tilde{p} = P_{F(T)} f(\tilde{p})$, which is a fixed point of T and, also, it is a solution of the variational inequality (3.2).

Since every Hilbert space is a complete CAT(0) space, from Corollary 3.4 we can obtain the following result immediately.

Corollary 3.5. *Let C be a nonempty closed and convex subset of real Hilbert space H and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(\beta_n x_n + (1 - \beta_n)x_{n+1})$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$ and satisfy the conditions (i)-(iii) as in Corollary 3.4. Then the sequence $\{x_n\}$ converges strongly to \tilde{p} such that $\tilde{p} = P_{F(T)} f(\tilde{p})$, which is a fixed point of T and, also, it is a solution of the variational inequality (3.5).

4. Numerical example

In this section, we will illustrate reckoning the convergence behavior of modified viscosity implicit iteration process (3.1) with numerical results for supporting main theorem.

Example 4.1. Let $X = \mathbb{R}$ be a Euclidean metric space, which is also a complete CAT(0) space and $C = [1, 4]$. Let $T : C \rightarrow C$ be defined by

$$Tx = \sqrt{x}.$$

It is obvious that T is an asymptotically nonexpansive mapping and a sequence $\{k_n = 1\}$ with $F(T) = \{1\}$.

Let $f : C \rightarrow C$ be defined by

$$fx = \sqrt{x}.$$

It is easy to see that f is contraction mapping.

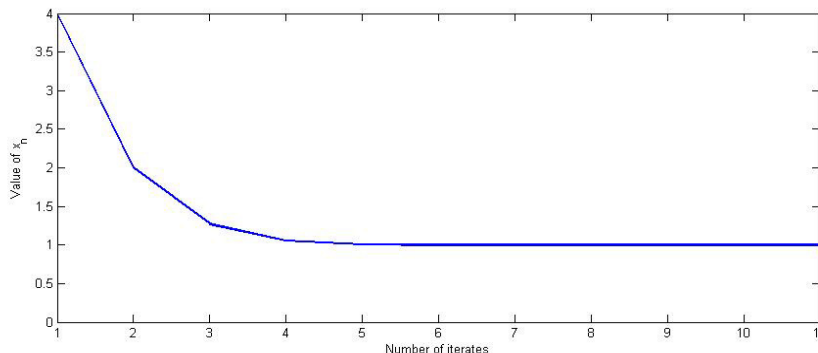


Figure 1: The value of x_n plotting in Table 1.

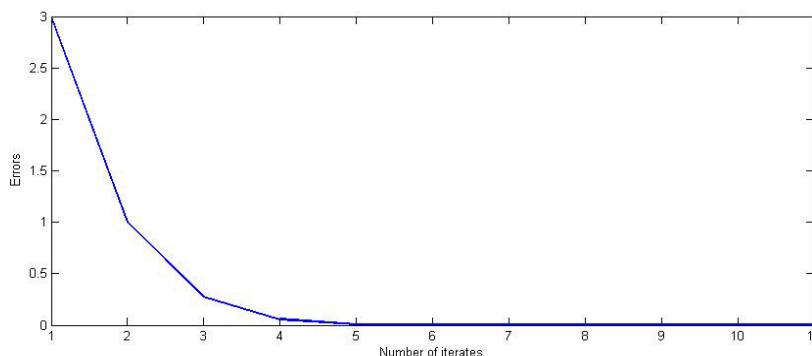


Figure 2: The value of $|x_n - p|$ (error) plotting in Table 1.

Table 1: The values of the sequence $\{x_n\}$ and the error values.

number of iterates	x_n	$ x_n - p $
1	4.000000000000	3.000000000000
2	2.000000000000	1.000000000000
3	1.272611350577	0.272611350577
4	1.055479646443	0.055479646443
5	1.008314617547	0.008314617547
6	1.000945233243	0.000945233243
7	1.000092259589	0.000092259589
8	1.000007257835	7.2578352E-06
9	1.000000479959	4.7995897E-07
10	1.000000028340	2.8339588E-08
11	1.000000001469	1.4694320E-09

Let $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{2}$ for all $n \geq 1$ and initial point $x_1 = 4$. Let $p = 1$ and $p \in F(T)$. Then we get numerical results in Table 1. Moreover, we also illustrate the convergence behavior of modified viscosity implicit iteration process (3.1) by the values of x_n as is shown in Figure 1. Figure 2 shows numerical results with an error (10^{-9}).

Next, we will show that the fixed point of T is solution of the following variational inequality:

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geq 0, \quad \forall p \in F(T),$$

where $\tilde{p} = P_{F(T)}f(\tilde{p})$.

Proof. Let $p \in F(T)$, and since $F(T) = \{1\}$ hence $p = 1$. By Theorem 3.1, we get

$$p = \tilde{p} = P_{F(T)}f(\tilde{p}).$$

For variational inequality, we have

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geq 0, \quad \langle (1) \cdot f(1), (1) \cdot (1) \rangle \geq 0, \quad \langle (1) \cdot (1), (1) \cdot (1) \rangle \geq 0.$$

Therefore the fixed point of T is a solution that satisfies the variational inequality. \square

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