Hybrid finite difference for solving differential equations

Prasert Phaochoo[1], Chainarong Wisassakwichai[2], Khanittha Promluang[3]

[1]Department of Mathematics, Faculty of Science and Technology, Rajamangala University of Technology Krungthep, Thailand.

[2]Department of Electrical Engineering, Faculty of Engineering, Rajamangala University of Technology Krungthep, Thailand.

[3]Department of Learning Management, Faculty of Education, Burapha University, Thailand.

*E-mail: prasert.p@mail.rmutk.ac.th, chainarong.w@rmutk.ac.th, k.promluang@gmail.com

ABSTRACT

A finite difference is widely used for solving numerical solution of differential equations. Some restriction of traditional finite difference affect to increasing an error of numerical solution. This paper presents a hybrid finite difference, upwind scheme and central finite difference for solving ordinary differential equations and partial differential equations which could have stability more than the traditional method. We found that the numerical solution by hybrid scheme get rid of maximum error more than central finite difference method and level of gamma parameter lead to decreasing of root mean square error.

INTRODUCTION

The finite difference method is used for solving ordinary differential equations and partial differential equations. Approximation of d(f(x))/dx is defined by forward difference approximation (FDA), backward difference approximation (BDA) or central difference approximation (CDA) in finite difference. The differentiable function of x can be expanded in a Taylor series about x. The idea of finite difference extend to a function of two variables for partial differential equation. BDA,FDA and CDA are replaced by u_x, u_y [1].

The upwind scheme is combination of backward finite difference and forward finite difference. The hybrid scheme is blended between upwind scheme and central finite difference. Configuration of gamma parameter will affect to level of upwind scheme and central finite difference in hybrid scheme. The upwind scheme is modified for convective-diffusion equations. A second order upwind scheme is applied for multidimensional magnetohydrodynamics in 1998 [2]. The Linear hyperbolic systems are discrete by second order upwind method [3]. The upwind scheme is represented to first order derivative and central finite difference is applied to second order derivative for change to elliptic problem [4]. The upwind compact scheme is solved with the Euler equation for the incompressible flow [5]. The OUCS2 upwind compact scheme is applied to calculation of first derivative in the Euler and Navier-Stokes equations [6]. Semi-Discrete Central scheme is constructed and analyzed by the total variation(TV) of approximation solution [7]. Central difference, upwind and hybrid scheme are solved in different grid system for general transport equation [8]. Triangular discretization of the domain is purposed with central upwind scheme for variable density shallow water flow equations [9]. The well-balanced positivity preserving second-order “triangular” central-upwind scheme is improved for the two-dimensional Saint-Venant system of shallow water equation [10].

In this paper, we propose a hybrid scheme for solving ordinary differential equations and partial differential equations. Numerical solutions are compared with analytic solutions in the different gamma parameter.

MATERIALS AND METHODS

This research use hybrid method for first derivative and central finite difference for second derivative. Blending of upwind scheme and central finite difference depend on gamma parameter. Configuration of gamma parameter in hybrid method is set as 0.1,0.5 and 0.9 respectively. Maximum norm and root mean square error (RMSE) are purposed for comparison numerical solution.

2.1 Ordinary differential equation.

We consider the ordinary differential equation as following

$$-\frac{d^2u}{dx^2}+k\frac{du}{dx}=\sin x. \qquad (1)$$

Boundary condition is $u(1)=2, u(3)=5.$

Analytic solution is

$$u(x)=c_1+c_2e^{kx}+\frac{\sin x-k\cos x}{1+k^2},$$

$$c_1=2-c_2e^k-\frac{\sin 1-k\cos 1}{1+k^2},$$

$$c_2=\left[3-\frac{(\sin 3-k\cos 3-\sin 1+k\cos 1)}{1+k^2}\right]\times\frac{1}{e^{3k}-e^k}\cdot$$

The Eq.(1) is demonstrated for comparision numerical solution between the central finite difference method and hybrid upwind scheme.

2.1.1 Central finite difference method for ODE

The Eq.(1) is discreted by central finite difference method as following

$$-\left.\frac{d^2u}{dx^2}\right|_{x_i}+k\left.\frac{du}{dx}\right|_{x_i}=f(x_i) \qquad (2)$$

$$-\left(\frac{u_{i-1}-2u_i+u_{i+1}}{(\Delta x)^2}\right)+k\left(\frac{u_{i+1}-u_{i-1}}{2\Delta x}\right)=f(x_i) \qquad (3)$$

$$-(k\Delta x+2)u_{i-1}+4u_i+(k\Delta x-2)u_{i+1}=2(\Delta x)^2 f(x_i) \qquad (4)$$

Let $L=-(k\Delta x+2).$

Let $R=(k\Delta x-2).$

Eq.(4) can be written in the matrix form as following

$$\begin{bmatrix} 4 & R & & & \\ L & 4 & R & & \\ & \ddots & \ddots & \ddots & \\ & & L & 4 & R \\ & & & L & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1}\end{bmatrix}=\begin{bmatrix} 2(\Delta x)^2 f(x_1)-Lu_0 \\ 2(\Delta x)^2 f(x_2) \\ \vdots \\ 2(\Delta x)^2 f(x_{n-1})-Ru_n\end{bmatrix} \qquad (5)$$

Let $A=\begin{bmatrix} 4 & R & & & \\ L & 4 & R & & \\ & \ddots & \ddots & \ddots & \\ & & L & 4 & R \\ & & & L & 4 \end{bmatrix}$

Let $U=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1}\end{bmatrix}$ and $F=\begin{bmatrix} 2(\Delta x)^2 f(x_1)-Lu_0 \\ 2(\Delta x)^2 f(x_2) \\ \vdots \\ \vdots \\ 2(\Delta x)^2 f(x_{n-1})-Ru_n\end{bmatrix}$

We rearrange (5) yields the following result $AU=F.$ Matrix U is solved by computer programming.

2.1.2 Hybrid upwind scheme for ODE

The Eq.(1) is discreted by hybrid upwind scheme as following

$$-\left.\frac{d^2u}{dx^2}\right|_{x_i}+k\left.\frac{du}{dx}\right|_{x_i}=f(x_i) \qquad (6)$$

$$\frac{d^2u}{dx^2}=\left(\frac{u_{i-1}-2u_i+u_{i+1}}{(\Delta x)^2}\right) \qquad (7)$$

$$\frac{du}{dx}=\frac{\gamma}{2\Delta x}\big((1+\varepsilon)(u_i-u_{i-1})+(1-\varepsilon)(u_{i+1}-u_i)\big)+\frac{(1-\gamma)}{2\Delta x}(u_{i+1}-u_{i-1}) \qquad (8)$$

Eq.(6) can be substitued by Eq.(7)-(8), the result as following

$$(-2-k\Delta x(\gamma\varepsilon+1))u_{i-1}+(4+2k\Delta x\gamma\varepsilon)u_i+(-2-k\Delta x(\gamma\varepsilon-1))u_{i+1}=2(\Delta x)^2 f(x_i) \qquad (9)$$

Let $L=-2-k\Delta x(\gamma\varepsilon+1).$

Let $C=4+2k\Delta x\gamma\varepsilon.$

Let $R=-2-k\Delta x(\gamma\varepsilon-1).$

Eq.(9) can be written in the matrix form as following

$$\begin{bmatrix} C & R & & & \\ L & C & R & & \\ & \ddots & \ddots & \ddots & \\ & & L & C & R \\ & & & L & C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} 2(\Delta x)^2 f(x_1) - Lu_0 \\ 2(\Delta x)^2 f(x_2) \\ \vdots \\ \\ 2(\Delta x)^2 f(x_{n-1}) - Ru_n \end{bmatrix}$$

2.2 Partial differential equation.

We consider the partial differential equation as following

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -2,\ 0 < x < 1, t > 0. \qquad (10)$$

Boundary condition is

Initial condition is $u(x,0) = x^2 + 1 \quad ,0 < x < 1.$

Analytic solution is $u(0,t) = 3, u(1,t) = 5, t > 0.$

$$u(x,t) = x^2 + x + 3 + \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}.$$

where $b_n = \begin{cases} \dfrac{2}{n\pi}, & n \text{ is even.} \\ \dfrac{-10}{n\pi}, & n \text{ is odd.} \end{cases}$

The analytic solution is an infinite series as

$$u(x,t) = x^2 + x + 3 - \frac{10}{\pi}\sin \pi x e^{-\pi^2 t} + \frac{1}{\pi}\sin 2\pi x e^{-4\pi^2 t} - \frac{10}{3\pi}\sin \pi x e^{-9\pi^2 t} + \frac{1}{2\pi}\sin 2\pi x e^{-16\pi^2 t} - \frac{2}{\pi}\sin \pi x e^{-25\pi^2 t} + \frac{1}{3\pi}\sin 2\pi x e^{-36\pi^2 t} - \ldots$$

2.2.1 Central finite difference method for PDE

The Eq.(10) is discreted by central finite difference method as following

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} \right) \qquad (11)$$

$$\frac{\partial u}{\partial t} = \frac{1}{2\Delta t}(u_{i+1} - u_{i-1}) \qquad (12)$$

Eq.(10) can be substitued by Eq.(11)-(12), the result as following

$$\left(-2(\Delta t) - (\Delta x)^2\right)u_{i-1} + \left(4(\Delta t)\right)u_i + \left(-2(\Delta t) + (\Delta x)^2\right)u_{i+1} = 2(\Delta t)(\Delta x)^2 f(x_i) \qquad (13)$$

Let $L = -2(\Delta t) - (\Delta x)^2$ is an coefficient of u_{i-1}.
Let $C = 4(\Delta t)$ is an coefficient of u_i.
Let $R = -2(\Delta t) + (\Delta x)^2$ is an coefficient of u_{i+1}.

We substitute L,C and R in the matrix form for solving numerical solution.

2.2.2 Hybrid upwind scheme for PDE

The Eq.(10) is discreted by hybrid upwind scheme as following

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} \right) \qquad (14)$$

$$\frac{\partial u}{\partial t} = \frac{\gamma}{2\Delta t}\left((1+\varepsilon)(u_i - u_{i-1}) + (1-\varepsilon)(u_{i+1} - u_i)\right) + \frac{(1-\gamma)}{2\Delta t}(u_{i+1} - u_{i-1}) \qquad (15)$$

Eq.(10) can be substitued by Eq.(14)-(15), the result as following

$$\left(-2(\Delta t) - (\Delta x)^2(\gamma\varepsilon + 1)\right)u_{i-1} + \left(4(\Delta t) + 2(\Delta x)^2 \gamma\varepsilon\right)u_i + \left(-2(\Delta t) - (\Delta x)^2(\varepsilon - \gamma + 2)\right)u_{i+1} = 2(\Delta t)(\Delta x)^2 f(x_i) \qquad (16)$$

Let $L = -2(\Delta t) - (\Delta x)^2(\gamma\varepsilon + 1)$ is an coefficient of u_{i-1}.
Let $C = 4(\Delta t) + (\Delta x)^2 \gamma\varepsilon$ is an coefficient of u_i.
Let $R = -2(\Delta t) + (\Delta x)^2(\varepsilon - \gamma + 2)$ is an coefficient of u_{i+1}.

We substitute L,C and R in the matrix form for solving numerical solution.

RESULTS AND DISCUSSION

This research will compare numerical solution of differential equation with central finite differential, hybrid upwind scheme and analytic solution.

Accuracy is measured in the discrete maximum norm and root mean square error (RMSE). The discrete maximum norm and maximum of root mean square error was given in Table 1-Table 3. that is estimated for difference gamma in hybrid scheme. The analytical and numerical solution profiles are given in Fig. 1- Fig. 11.

$$\text{Maximum Norm} \quad = max\left|\hat{\boldsymbol{u}}_i - \boldsymbol{u}_i\right|$$

$$\text{RMSE} \quad = \sqrt{\frac{1}{\boldsymbol{N}}\sum_{i=1}^{N}\left(\hat{\boldsymbol{u}}_i - \boldsymbol{u}_i\right)^2}$$

where $\hat{\boldsymbol{u}}_i$ is an approximate solution of differential equation and u_i is the analytic solution .

Example 1. We consider the ordinary differential equation $-\frac{d^2u}{dx^2} + k\frac{du}{dx} = 0$.

The analytic solution is $u(x) = c_1 + c_2 e^{kx}$ where

$c_1 = -c_2 e^k, \quad c_2 = \frac{1}{1-e^k}$.

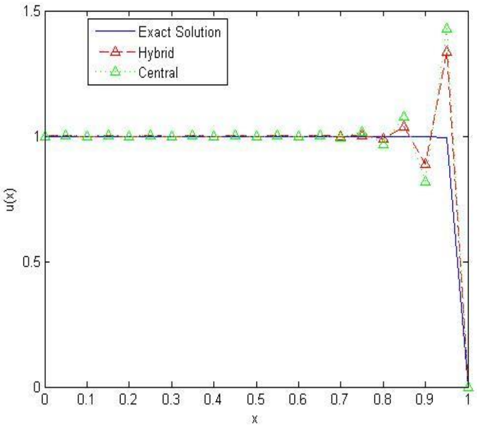

Figure 1. Numerical solutions with parameter $\gamma = 0.1$ of hybrid method.

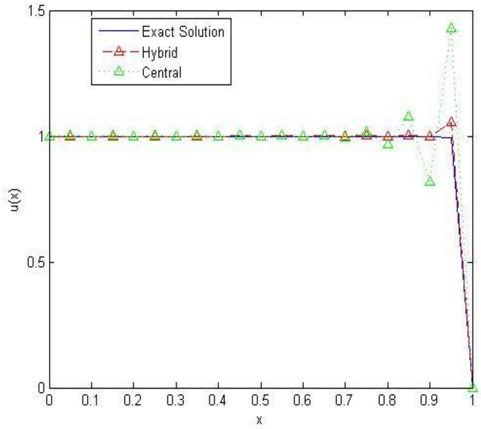

Figure 2. Numerical solutions with parameter $\gamma = 0.5$ of hybrid method.

Table 1. The numerical solutions by difference gamma in Example 1.

Method	Central	Hybrid	
Error [a]		$\gamma = 0.1$	$\gamma = 0.5$
MaxNorm	0.435309	0.340071	0.059370
RMSE	0.151922	0.113810	0.018794

Example 2. We consider the ordinary differential equation $-\frac{d^2u}{dx^2} + k\frac{du}{dx} = \sin x$.

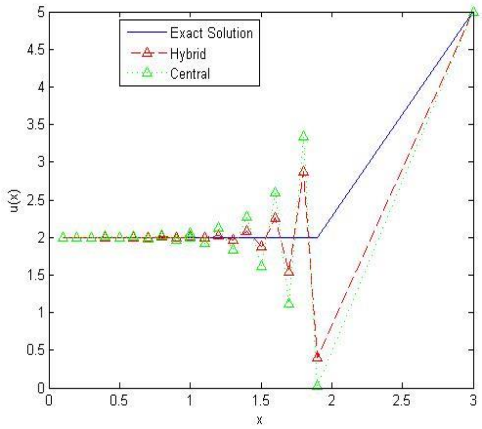

Figure 3. Numerical solutions with parameter $\gamma = 0.1$ of hybrid method.

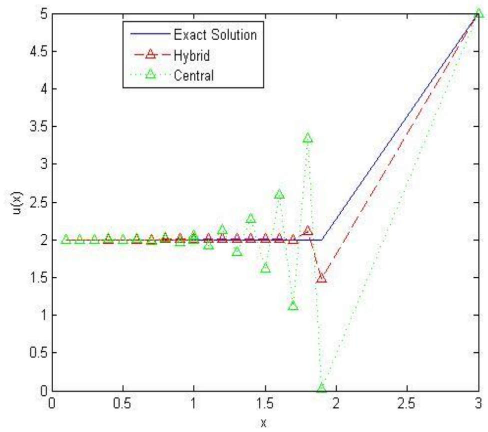

Figure 4. Numerical solutions with parameter $\gamma = 0.5$ of hybrid method.

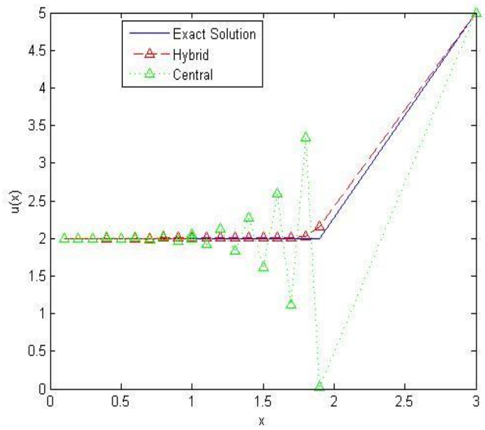

Figure 5. Numerical solutions with parameter $\gamma = 0.9$ of hybrid method.

Table 2. The numerical solutions by difference gamma in Example 2.

Method	Central	Hybrid		
Error [a]		$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.9$
MaxNorm	1.978637	1.594305	0.513291	0.155889
RMSE	0.842973	0.601025	0.165887	0.050337

Example 3. We consider the partial differential equation $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = -2,\ 0 < x < 1,\ t > 0.$

Boundary Condition is $u(0,t) = 3, u(1,t) = 5 \quad ,t > 0.$

Initial Condition is $u(x,0) = x^2 + 1 \quad ,0 < x < 1.$

The analytic solution is

$$u(x,t) = x^2 + x + 3 - \frac{10}{\pi}\sin \pi x e^{-\pi^2 t} + \frac{1}{\pi}\sin 2\pi x e^{-4\pi^2 t} + \ldots$$

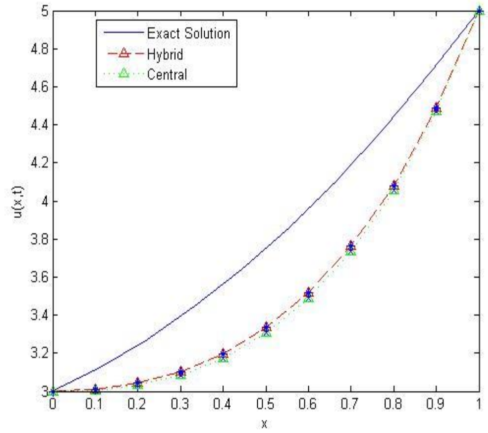

Figure 6. Numerical solutions with parameter $\gamma = 0.5$ of hybrid method.

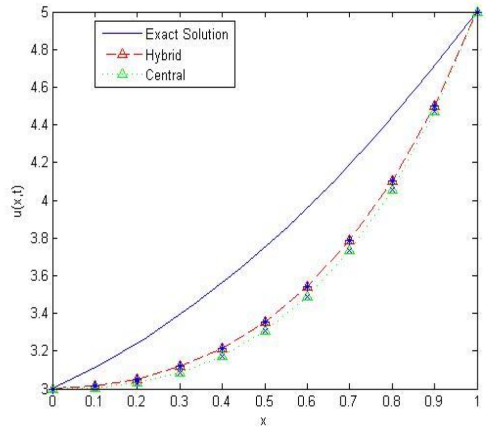

Figure 7. Numerical solutions with parameter $\gamma = 0.9$ of hybrid method.

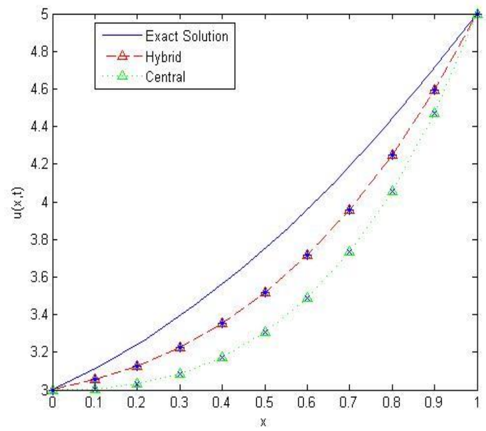

Figure 8. Numerical solutions with parameter $\gamma = 5$ of hybrid method.

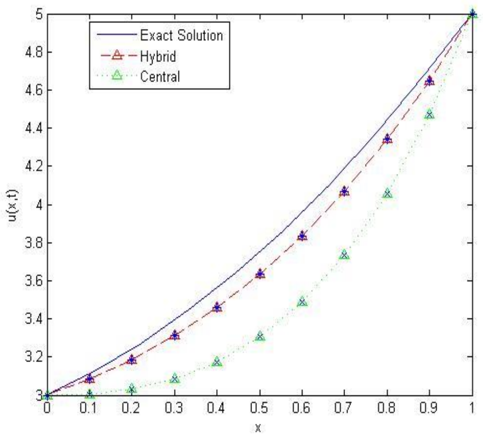

Figure 9. Numerical solutions with parameter $\gamma = 10$ of hybrid method.

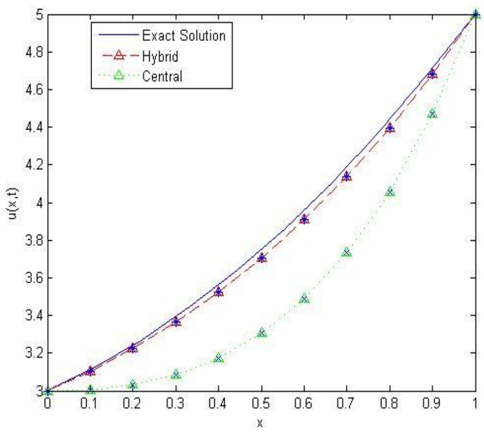

Figure 10. Numerical solutions with parameter $\gamma = 15$ of hybrid method.

Table 3. The numerical solutions by difference gamma in Example 3.

Method	Central	Hybrid		
Error [a]		$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.9$
MaxNorm	0.410521	0.379110	0.182407	0.064411
RMSE	0.286993	0.264867	0.126159	0.024845

CONCLUSIONS

The hybrid scheme has maximum norm and root mean square error less than central finite difference method in addition to the most of maximum gamma will have lower error for same hybrid scheme. Increasing of gamma parameter in partial differential equation will decrease maximum norm and root mean square error. Moreover, convergence rate will correspond with gamma parameter in hybrid scheme.

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