Fractional calculus by inverse Laplace transform for analysis radioactivity

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ABSTRACT

A fractional calculus is an excellent tool for solving problems in many researches. This paper presents a fractional calculus for solving differential equations in the problem of radioactive decay. The method for solving problem is inverse Laplace transform of fractional calculus that concerns with Riemann-Liouville fractional derivative, Riemann-Liouville fractional Integral, Mittag Leffler function and Gamma function. The numerical solution of fractional calculus will give the value of radioactivity less than the traditional calculus.

INTRODUCTION

The first idea of fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1965 about derivative with non-integer. The Fractional calculus is a name for the theory of derivatives and integrals of arbitrary order. The definition of derivative is described by Euler in 1730 and Laplace in 1812. Lacroix is the French mathematics who presented a derivative of non-integer in term of Legendre's symbol Γ . Lacroix expresses n^{th} derivative of function $y = x^m$ as follows [1]

$$\frac{d^n y}{dx^n} y = \frac{m!}{(m-n)!} x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

By letting $m = 1, n = \frac{1}{2}$, he obtained

$$\frac{d^{1/2} y}{dx^{1/2}} y = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

The first application of fractional calculus is concerned with solution of the integral equation to the tautochronous problem that presented by Abel in 1823[2]. He founded that the solution of this problem could be formed as a semi-derivative. Liouville was the major study of many mathematicians defined and developed the formula of fractional integral and fractional derivative. Riemann derived different definition that concerned a definite integral by adding the complementary function in 1853. Caputo found that the value of certain fractional integrals and derivative need to be specified at the initial time for the solution of fractional differential equation and solved some problems of viscoelasticity in 1967 [3]. Nowadays, the definition of the Riemann-Liouville has been popularized in the world of fractional calculus.

Fractional differential equation have attracted much attention during the past few decade. This is the fact that fractional calculus supply an competent and excellent tool for the description of many important phenomena such as electromagnetic, physics, chemistry, biology, economy and many more.

Fractional calculus is used to describe about ultrasonic wave propagation in human cancellous bone [4], speech signal modeling [5], cardiac tissue electrode interface [6], path-tracking problem in an autonomous electric vehicle [7], theory of viscoelasticity [8], diffusion fluid mechanics problem [9], edge detection of image processing [10], RLC electrical circuit [11], relaxation and oscillation equation [12], mortgage problem in financial business [13] and Black-Scholes equation for option pricing [14].

In this paper, we propose a fractional calculus for solving differential equations in the problem of radioactive decay by using the inverse Laplace transform of fractional calculus.

MATERIALS AND METHODS

The formula of inverse Laplace transform is solved with Riemann-Liouville fractional derivative, Riemann-Liouville fractional Integral and Mittag-Leffler function meanwhile Mittag-Leffler function is derived from Gamma function and related with Error function.

2.1The Gamma function is a generalization the fractional of the factorial function, denoted by $\Gamma(x)$.

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\quad x\in\mathbb{R}^+.$$

2.2 The Error function is given by

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt,\quad x\in\mathbb{R}.$$

The Complementary error function (erfc) can be written in form of the Error function as:

$$erfc(x)=1-erf(x).$$

2.3 The Mittag Leffler function is a major function in fractional calculus was defined by Mittag-Leffler in 1903. The Mittag Leffler function can be defined in term of a power series as follow :

$$E_\alpha(x)=\sum_{k=1}^\infty\frac{x^k}{\Gamma(\alpha k+1)},\quad \alpha>0.$$

The two parameter α and β are defined in the Mittag Leffler function as follow:

$$E_{\alpha,\beta}(x)=\sum_{k=0}^\infty\frac{x^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\beta>0.$$

The example of the Mittag Leffler function such as :

$$E_{1,1}(x)=e^x,$$
$$E_{2,1}(x^2)=\cosh(x),$$
$$E_{2,2}(x^2)=\frac{\sinh(x)}{x},$$
$$E_{1/2,1}(x)=e^{x^2}erfc(-x).$$

2.4 The Riemann-Liouville fractional Integral of function f(x) of order ν as:

$${}_cD_x^{-\nu}f(x)=\frac{1}{\Gamma(\nu)}\int_c^x(x-t)^{\nu-1}f(t)dt,\quad \nu>0.$$

$${}_cD_x^{-n}f(x)=\frac{1}{(n-1)!}\int_c^x(x-t)^{n-1}f(t)dt.$$

We can solve the Riemann-Liouville fractional Integral as:

$$D^{-\nu}x^\mu=\frac{\Gamma(\mu+1)}{(\mu+\nu+1)}x^{\mu+\nu},\nu>0,\mu>-1,x>0.$$

Suppose the functions are selected by transcendental functions such as $f(t)=e^{at}$ that fractional Integral as:

$$D^{-\nu}e^{at}=E_t(\nu,a)=t^\nu E_{1,\nu+1}(at).$$

2.5 The Riemann-Liouville fractional derivative function f(x) of order ν will denote by notation ${}_cD_x^\nu f(x)$, $\nu>0$. We often drop subscript c and x.

$$D^u f(x)=D^n\left[D^{-\nu}f(x)\right]$$

The derivative of function $f(x)=x^\mu$ of order $\nu,\mu\geq 0$ can be changed by setting $u=n-\nu$ where $0<u<1$.

$$D^{\nu} f(x) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)} x^{\mu-\nu}, \mu \geq 0, 0 < \nu < 1.$$

2.6 The Laplace transform of the fractional integral.

The original Laplace transform of y(t) as follows :

$$Y(s) = L\{y(t)\} = \int_0^{\infty} y(t) e^{-st} dt,$$

$y(t) = L^{-1}\{Y(s)\}$ is the inverse Laplace transform of $Y(s)$.

The Laplace transform of the fractional integral of $y(t)$ order ν is defined as [15],

$$D^{-\nu} y(t) = \frac{1}{\Gamma(\nu)} \int_c^t (t-z)^{\nu-1} y(z) dz, \quad \nu > 0.$$

$$L\{D^{-\nu} y(t)\} = \frac{1}{\Gamma(\nu)} L\{t^{\nu-1}\} L\{y(t)\}, \quad \nu > 0.$$

$$= s^{-\nu} y(s), \quad \nu > 0.$$

RESULTS AND DISCUSSION

This research will forecast the residue in the future. We simulate data for radioactive decay rate at 5 and 10 percent while peroid of time for decay are 100 and 300 years. The first time, we predict the value of radioactivity by traditional calculus. Next step, four examples are purposed for comparison the value of radioactivity in other situations. The derivative order in original calculus is 1 whereas the fractional derivative order are $8/9, 2/3$ and $1/2$ respectively.

Example 1. Suppose that radioactive decay rates are proportion with quantity of radioactivity at present and decay rate is 10 percent per 100 years. Scientist would like to predict the value of radioactivity in the future.

For first time, we use ordinary differential equation and let x is a residue at t year then $\frac{dx}{dt} = -kx$, k is constant of proportion. The solution by separable variable method is $x = ce^{-kt}$, c is constant.

Let x_0 is initial value then $c = x_0$ and $x = x_0 e^{-kt}$. If decay rate is 10 percent per 100 years then $e^{-k} = (0.9)^{0.01}$, $x = x_0 (0.9)^{(0.01)t}$ and the percent of residue for next 1000 years is 34.87%.

In this case, if a quantity of residue is one fourth of initial value then time for this situation will be 1,316 years.

Next step, we use the fractional differential equation with same problem and define the fractional derivative order are $8/9, 2/3$ and $1/2$ respectively.

$$D^{\frac{2}{3}} x(t) = kx(t), 0 < \nu < 1, \nu = \frac{2}{3}.$$

$$L\left\{D^{\frac{2}{3}} x(t)\right\} = kL\{x(t)\},$$

$$s^{\frac{2}{3}} Y(s) - D^{-\left(1-\frac{2}{3}\right)} y(0) = kY(s).$$

Assume that $c_1 = D^{-\frac{1}{3}} y(0)$ then

$$s^{\frac{2}{3}} Y(s) - c_1 = kY(s),$$

$$Y(s) = \frac{c_1}{s^{\frac{2}{3}} - k}.$$

We take fractional inverse Laplace of $Y(s)$ then

$$y(t) = L^{-1}\left\{\frac{c_1}{s^{\frac{2}{3}} - k}\right\} = c_1 t^{-\frac{1}{3}} E_{2/3,2/3}\left(at^{\frac{2}{3}}\right)$$

We imply that

$$D^{\nu} x(t) = kx(t),$$

$$L\left\{D^{\nu}x(t)\right\}=kL\left\{x(t)\right\},$$
$$s^{\nu}Y(s)-D^{-(1-\nu)}y(0)=kY(s).$$

Assume that $c_1=D^{-(1-\nu)}y(0)$ then

$$s^{o}Y(s)-c_1=kY(s)$$
$$Y(s)=\frac{c_1}{s^{\nu}-k}$$

We take fractional inverse Laplace of $Y(s)$ then

$$y(t)=L^{-1}\left\{\frac{c_1}{s^{\nu}-k}\right\}=c_1t^{\nu-1}E_{\nu,\nu}\left(at^{\nu}\right).$$

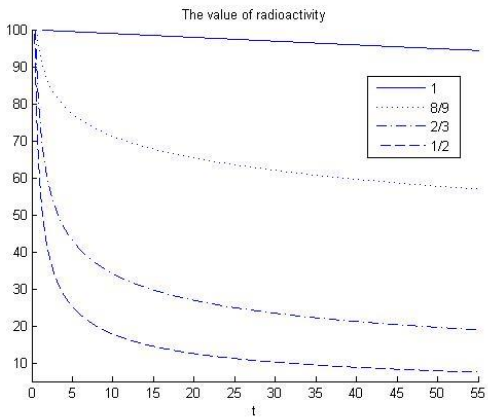

Figure 1. The comparison numerical solutions of radioactive decay at 10 percent per 100 years by fractional differential equation in derivative order $8/9, 2/3, 1/2$ with original differential equation (order 1).

Example 2. Suppose that radioactive decay rate is 10 percent per 300 years.

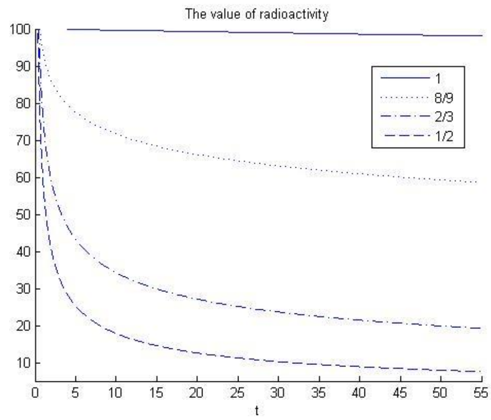

Figure 2. The comparison numerical solutions of radioactive decay at 10 percent per 300 years by fractional differential equation in derivative order $8/9, 2/3, 1/2$ with original differential equation (order 1).

Example 3. Suppose that radioactive decay rate is 5 percent per 100 years.

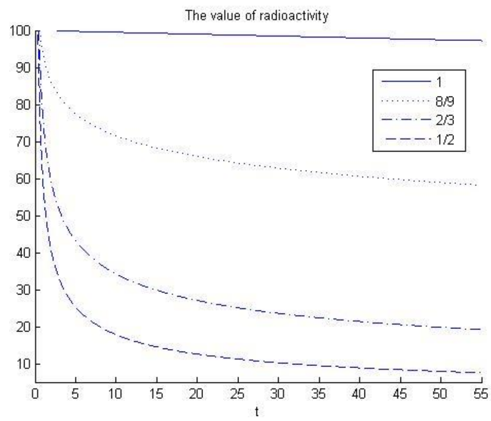

Figure 3. The comparison numerical solutions of radioactive decay at 5 percent per 100 years by fractional differential equation in derivative order $8/9, 2/3, 1/2$ with original differential equation (order 1).

Example 4. Suppose that radioactive decay is 5 percent per 300 years.

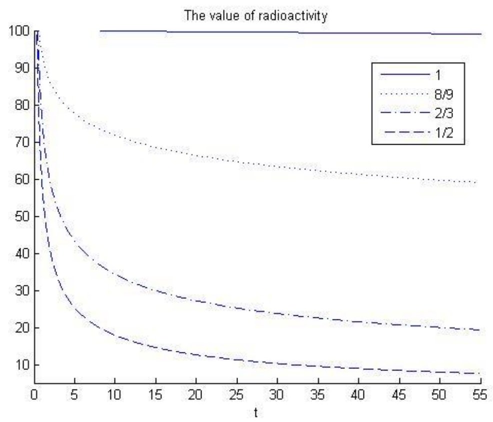

Figure 4. The comparison numerical solutions of radioactive decay at 5 percent per 300 years by fractional differential equation in derivative order $8/9, 2/3, 1/2$ with original differential equation (order 1).

We found that the value of radioactivity for fractional derivative order less than integer derivative order (order 1) in all cases. In case of fractional derivative order, higher order will give value of radioactivity more than lower order when comparison occur at the same time. Data of Table 1. to Table 4. present value of radioactivity in details for all cases from Example 1 to Example 4.

Table 1. The numerical solutions of integer derivative order for the radioactive decay at 10 percent per 100 years compare with solution of fractional derivative order from 10 to 40 years.

Order	1	8/9	2/3	1/2
Year[a]		Residue[b]		
10	98.95	71.13	33.98	17.71
20	97.91	65.33	26.86	12.49
30	96.88	61.97	23.38	10.18
40	95.87	59.61	21.19	8.81

Table 2. The numerical solutions of integer derivative order for the radioactive decay at 10 percent per 300 years compare with solution of fractional derivative order from 10 to 40 years.

Order	1	8/9	2/3	1/2
Year[a]		Residue[b]		
10	99.65	71.58	34.15	17.78
20	99.30	66.10	27.07	12.56
30	98.95	63.03	23.62	10.25
40	98.60	60.92	21.46	8.88

Table 3. The numerical solutions of integer derivative order for the radioactive decay at 5 percent per 100 years compare with solution of fractional derivative order from 10 to 40 years.

Order	1	8/9	2/3	1/2
Year[a]		Residue[b]		
10	99.46	71.48	34.11	17.76
20	98.98	65.92	27.02	12.54
30	98.47	62.78	23.56	10.23
40	97.97	60.61	21.40	8.86

Table 4. The numerical solutions of integer derivative order for the radioactive decay at 5 percent per 300 years compare with solution of fractional derivative order from 10 to 40 years.

Order	1	8/9	2/3	1/2
Year[a]		Residue[b]		
10	99.83	71.70	34.20	17.79
20	99.66	66.30	27.12	12.58
30	99.49	63.30	23.68	10.27
40	95.32	61.26	21.53	8.90

[a]The unit of year is year.
[b] The unit of residue is ton.

CONCLUSIONS

In this paper, the fractional differential equation is solved with the inverse Laplace transform of fractional calculus. The value of radioactivity by traditional differential equation is compared with the fractional differential equation.

The numerical solutions of every example present results for varieties of orders and the percentage of radioactive decay. We found that the every time has a difference for the value of radioactivity in every derivative order. The value will be slightly difference between an initial time and last time for integer derivative order however case of the fractional derivative order have more different. The fractional derivative order have the value of radioactivity less than integer derivative order (order 1) while the higher order of fractional calculus will give value of radioactivity more than the lower order of fractional calculus. This solutions can use for management the residue in the future.

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