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COINCIDENCE POINT THEOREMS OF MULTIDIMENSIONAL FOR (φ)-WEAK CONTRACTION IN FUZZY METRIC SPACES

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Abstract

In this paper, we use the notion of (φ)-weak contraction to study the existence of multidimensional coincidence points on fuzzy metric spaces in the sense of GV [11]. Then, coincidence point results for two mappings are obtained.

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1. Introduction

Fixed point theory is important in the study of functional analysis. In particular, fixed point theory applied to the field of Economics, Computer Science, Biology, Chemistry, and Engineering. The notion of coupled fixed point was given by in 1987. After this work, Choudhury and Kundu [4] define the notion of coupled coincidence point result in partially ordered metric spaces for compatible mappings. Following these many authors proved coincidence point theorems including [4, 20, 28, 29]. The concept of tripled fixed point was given by Berinde and Borcut [3] which is generalizing the notion of a coupled fixed point to a tripled fixed point. Karapinar [17] improved this idea by defining quadruple fixed point and n-tuplet fixed point. Very recently, Roldan et al. [26] generalized this idea by introducing multidimensional fixed point.

Kramosil and Michàlet [19] were first studied the concept of fuzzy metric spaces. After that, George and Veeramani [7] were modified the concept of fuzzy metric spaces of Kramosil's. In 1998, Grabiec [9] extend fixed point theorems of fuzzy metric spaces in the sense of Kramosil and Michalet. In 2000, Gregori et al. [10] introduced a notion of precompact fuzzy metric space which provides several satisfactory and presented examples of fuzzy metrics in the sense of Grorge and Veeramani [7]. In 2000, Mihet [17] studied fixed point theorem in fuzzy metric spaces by using E.A. property. Afterwards, in 2014, Roldan et al. [26] studied and modified multidimensional coincidence point results in fuzzy metric spaces for a fuzzy contractivity condition.

In this paper, the concept of (φ)- weak contractions is introduced in fuzzy metric spaces. Consequently existence and coincidence point for such mappings are obtained. Our results substantially generalize and improve several comparable results in the literature (see [7], [8], [23]).

Henceforth, we gave some preliminaries definitions and main results in fuzzy metric spaces.

2. Preliminaries

2.1. Triangular norms

Definition 2.1 ([27]). A *continuous t-norm* is a binary operation $\Delta : [0, 1]^2 \to [0, 1]$, if it satisfies the following conditions:

(1) $a \Delta b = b \Delta a$,

(2) $(a \Delta b) \Delta c = a \Delta (b \Delta c)$,

(3) Δ is continuous,

(4) $1 \Delta a = a$ for all $a \in [0, 1]$,

(5) $a \Delta b \leq c \Delta d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

If $a_1, a_2, \ldots, a_m \in [0, 1]$, then $\Delta_{i=1}^{m} a_i = a_1 \Delta a_2 \Delta \ldots \Delta a_m$.

For each $a \in [0, 1]$, the sequence $\{\Delta^m\}_{m=1}^{\infty}$ is defined inductively by $\Delta^1 a = a$ and $\Delta^{m+1} a = (\Delta^m a) \Delta a$ for all $m \geq 1$.

Definition 2.2 ([14]). Let $\sup_{0<a<1} \Delta(a, a) = 1$. A continuous t-norm Δ is said to be of Hadžić-type (H-type) if the family of functions $\{\Delta^m(a)\}_{m=1}^{\infty}$ is equi-continuous at $a = 1$. The t-norm $\Delta_M := \min$ is an example of H-type.

Other examples can be found in [12]. Examples of continuous t-norms are Lukasievicz t-norm, that is, $a \Delta_L b = \max\{a + b - 1, 0\}$, product t-norm, that is, $a \Delta_P b = ab$ and minimum t-norm, that is, $a \Delta_M b = \min\{a, b\}$.

2.2. Fuzzy metric spaces

We shall study the basic definitions and properties about fuzzy metric spaces. The concept of fuzzy metric spaces was given by George and Veeramani [7] as follows.

Definition 2.3 ([7]). Let X be a nonempty set, Δ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$. The 3-tuple (X, M, Δ) is said to be a *fuzzy metric space* if satisfying the following conditions, for every $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ iff $x = y$ for all $t > 0$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t+s) \geq M(x, z, t) \Delta M(z, y, s)$ for all $t, s > 0$,

(v) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In this case, we also say that (X, M, Δ) is a fuzzy metric space under Δ. Consequently, we will only consider fuzzy metric space satisfying :

(vi) $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$.

Remark 2.4. We note that $0 < M(x, y, t) < 1$ (for all $t > 0$) provided $x \neq y$, (see [17]).

Let (X, M, Δ) be a fuzzy metric space. For $t > 0$, the open ball $S(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$S(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Set A is called open if $A \subset X$ for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $S(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X, we say the topology induced by the fuzzy metric M. This topology is metrizable (see in [10]).

We note that $M(x, y, t)$ be a definition of nearness between x and y with respect to t. Besides, it is known that $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

Definition 2.5 ([31])**.** Let (X, M, Δ) be a fuzzy metric space. A sequence $\{x_n\}$ in X is a Cauchy sequence if for any $0 < \varepsilon < 1$ and for all $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for any $n, m \geq n_0$.

Let $\{x_n\}$ be a sequence in X, that is Cauchy sequence in the sense of Grabiec [9] if $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$, for any $p \in \mathbb{N}$ and for all $t > 0$. If every Cauchy sequence is convergent, then fuzzy metric space (X, M, Δ) is complete. Now we show an example of fuzzy metric spaces defined by Gregori et al. [11].

Example 2.6 ([11])**.** Let (X, d) be a metric space and $g : \mathbb{R}^+ \to [0, \infty)$ be an increasing continuous function. Define $M : X^2 \times (0, \infty) \to [0, 1]$ as

$$M(x, y, t) = e^{\left(-\frac{d(x, y)}{g(t)}\right)},$$

for each $x, y \in X$ and for every $t > 0$. Then (X, M, Δ) is a fuzzy metric space on X, where Δ is the product t-norm.

Remark 2.7 ([14])**.** Let (X, d) be a metric space. Define $a \Delta b = ab$ for any $a, b \in [0, 1]$ and $M_d(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}$ for any $(x, y, t) \in X^2 \times [0, \infty)$, where k, m and n are positive real numbers then (X, M_d, Δ) is said to be a *fuzzy metric space*. Thus, every metric space induces a fuzzy metric space. The fuzzy metric given by $M_d(x, y, t) = \frac{t}{t + d(x, y)}$ for any $(x, y, t) \in X^2 \times [0, \infty)$ is said to be *standard fuzzy metric*.

Lemma 2.8 ([9])**.** *Let* (X, M, Δ) *be a fuzzy metric space. For all* $x, y \in X$, $M(x, y, \cdot)$ *is non-decreasing function*.

Definition 2.9. Let (X, M, Δ) be a fuzzy metric space. Then the mapping M on $X^2 \times (0, \infty)$ is called continuous if

$$\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t),$$

where $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, that is,

$$\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 2.10 ([24])**.** *If* (X, M, Δ) *be a fuzzy metric space*, *then* M *is a continuous function on* $X^2 \times (0, \infty)$.

Definition 2.11 ([26])**.** Let $p \in \mathbb{N}$ and let (X, M, Δ) be a fuzzy metric space. A mapping $G : X^m \to X$ is called continuous at a point $\mathbf{y}_0 \in X^m$ if for every sequence $\{\mathbf{y}_n\}_{n\geq 0}$ in X^m converging to $\mathbf{y}_0$, the sequence $\{G(\mathbf{y}_n)\}_{n\geq 0}$ converges to $G(\mathbf{y}_0)$. If G is continuous at each $\mathbf{y}_0 \in X^m$, then G is called continuous on X^m.

2.3. The product spaces and coincidence points

Let X be a nonempty set. We consider the product set

$$X^m = \underbrace{X \times X \times \ldots \times X}_{m \text{ times}}.$$

The vectorial notation $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is the element of X^m, where $x_j \in X$ for $j = 1, \ldots, m$. In the sequel, let the function $T : X^m \to X$ defined by

$$T\mathbf{x} = T(x_1, x_2, \ldots, x_m).$$

Let X be a nonempty set. We consider the functions $T : X^m \to X$ and $f : X \to X$ satisfying $T(X^m) \subseteq f(X)$. Given an initial element $\mathbf{x}_0 = (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(m)}) \in X^m$, where $x_0^{(j)} \in X$ for $j = 1, \ldots, m$, since $T(X^m) \subseteq f(X)$, there exists $x_1^{(j)} \in X$ such that $f(x_1^{(j)}) = T(\mathbf{x}_0)$. Similarly, there also exists $x_2^{(j)} \in X^m$ such that $f(x_2^{(j)}) = T(\mathbf{x}_1)$. Continuing this process, we can construct a sequence $\{x_n^{(j)}\}_{n\in\mathbb{N}}$ such that

$$f(x_n^{(j)} = T(\mathbf{x}_{n-1}) \tag{1}$$

for all $n \in \mathbb{N}$ and for $j = 1, \ldots, m$.

We defined the concept of coincidence points and common fixed point as follows:

Definition 2.12. Let X be a nonempty set and $T : X^m \to X$ and $f : X \to X$ be two mappings. A point $(x_1, x_2, \ldots, x_m) \in X^m$ is said to be a *coincidence point of* T and f if $T(x_1, x_2, \ldots, x_n) = fx^{(j)}(T\mathbf{x} = fx^{(j)})$ and is said to be a *common fixed point of* T and f if $fx^{(j)} = T\mathbf{x} = x^{(j)}$ for

$j = 1, \ldots, m$. If f is the identity mapping on X, then $(x_1, x_2, \ldots, x_n) \in X^m$ is said to be a *fixed point* of the mapping T.

Remark 2.13 [26]**.** If T and f are commuting and $(x_1, x_2, \ldots, x_n) \in X^m$ is a coincidence point of T and f, then $(fx_1, fx_2, \ldots, fx_n)$ is also a coincidence point of T and f.

Definition 2.14. The mapping T and f are commuting if $fT(x_1, x_2, \ldots, x_m) = T(fx_1, fx_2, \ldots, fx_m)$ for all $x_1, x_2, \ldots, x_m \in X$.

From the concept ψ-weak contraction in [1], we defined the definition of (φ)- weak contraction mappings in product space of fuzzy metric spaces.

Definition 2.15. Let (X, M, Δ) be a fuzzy metric space and the mapping $f : X \to X$ be a self mapping. The mapping $T : X^m \to X$ is said to be a (φ)- weak contraction with respect to f if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(r) > 0$ for $r > 0$, $\varphi(0) = 0$, $\varphi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$ and $\mathbf{x}, \mathbf{y} \in X^m$ such that

$$\frac{1}{M(T\mathbf{x}, T\mathbf{y}, t)} - 1 \leq \left(\frac{1}{M(fx^{(i)}, fy^{(i)}, t)} - 1\right) - \varphi\left(\frac{1}{M(fx^{(i)}, fy^{(i)}, t)} - 1\right) \tag{2}$$

holds for every $x, y \in X$ and each $t > 0$. If the mapping f is the identity mapping, then the mapping $T : X^m \to X$ is said to be a (φ)- weak contraction.

3. Main Results

Theorem 3.1. *Let X be a nonempty set. Suppose that (X, M, Δ) is a complete fuzzy metric space with Δ a t-norm (H-type), $f : X \to X$ be a (φ)- weak contraction on X. If there exist $x_0 \in X$, the sequence $\{x_n\}_{n \geq 0}$ given by $f(x_n) = x_{n+1}$ has a convergent subsequence, then f has a fixed point.*

Proof. We consider $\{x_{n_i}\}_{n \in \mathbb{N}}$ be a convergent subsequence of $\{x_n\}$ with constructed (2) and $\{x_n\}$ is converges to x in X. Therefore there exists $t > 0$ such that

$$\lim_{i \to \infty} M(x_{n_i}, x, t) = 1. \tag{3}$$

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we can find $t > 0$ such that $M(x_1, x_0, t) > 0$ and $M(x_1, x_0, t) \to 1$. By (2), for $n = 1$ we have

$$\frac{1}{M(x_2, x_1, t)} - 1 = \left(\frac{1}{M(fx_1, fx_0, t)} - 1\right)$$

$$\leq \left(\frac{1}{M(x_1, x_0, t)} - 1\right) - \varphi\left(\frac{1}{M(x_1, x_0, t)} - 1\right),$$

and $n = 2$ we have

$$\frac{1}{M(x_3, x_2, t)} - 1 = \left(\frac{1}{M(fx_2, fx_1, t)} - 1\right)$$

$$\leq \left(\frac{1}{M(x_2, x_1, t)} - 1\right) - \varphi^2\left(\frac{1}{M(x_1, x_0, t)} - 1\right)$$

$$\leq \left(\frac{1}{M(x_1, x_0, t)} - 1\right) - \varphi^2\left(\frac{1}{M(x_1, x_0, t)} - 1\right) - \varphi\left(\frac{1}{M(x_1, x_0, t)} - 1\right).$$

Repeating the above procedure successively n times we have

$$\frac{1}{M(x_{n+1}, x_n, t)} - 1 = \left(\frac{1}{M(fx_n, fx_{n-1}, t)} - 1\right)$$

$$\leq \left(\frac{1}{M(x_1, x_0, t)} - 1\right) - \varphi^n\left(\frac{1}{M(x_1, x_0, t)} - 1\right)$$

$$- \ldots - \varphi\left(\frac{1}{M(x_1, x_0, t)} - 1\right).$$

Since $\varphi^n(a_n) \to 0$ whenever $a_n \to 0$, we have $\lim_{i \to \infty} M(x_{n+1}, x_n, t) = 1$, such that

$$\lim_{i \to \infty} M(x_{i+1}, x_{n_i}, t) = 1.$$

Given $\epsilon, \delta \in (0, 1)$ there exist i_1, i_2 in $\mathbb{N}$ such that for any $i' > i_1$ and $i'' > i_2$, we have

$$M(x_{n_{i'}}, x, t) > 1 - \delta$$

and

$$M(x_{n_{i''}+1}, x_{n_{i''}}, t) = 1 - \delta.$$

Take $i_0 = \max\{i', i''\}$ for any $j > i_0$, we have

$$M(x_{n_j}, x, t) > 1 - \delta$$

and

$$M(x_{n_j+1}, x_{n_j}, t) = 1 - \delta. \tag{4}$$

Therefore, we obtain

$$\begin{aligned} M(x_{n_j+1}, x, t) &\geq M(x_{n_j+1}, x_{n_j}, \frac{t}{2}) \Delta M(x_{n_j}, x, \frac{t}{2}) \\ &\geq (1-\delta) \Delta (1-\delta) \\ &= 1 \text{ (by } \Delta \text{ is a } H\text{-type)} \\ &\geq 1 - \epsilon. \end{aligned}$$

So, we get

$$\lim_{j \to \infty} M(x_{n_j+1}, x, t) = 1. \tag{5}$$

Further, from (4) and (5) it is possible to find $n_0 \in \mathbb{N}$ such that for any $j > n_0$

$$M(x_{n_j+1}, x, t) > 0.$$

Therefore, for each $j > n_0$, we have

$$\begin{aligned} \frac{1}{M(x_{n_j+1}, fx, t)} - 1 &\leq \left(\frac{1}{M(fx_{n_j}, fx, t)} - 1 \right) \\ &\leq \left(\frac{1}{M(x_{n_j}, x, t)} - 1 \right) - \varphi\left(\frac{1}{M(x_{n_j}, x, t)} - 1 \right) \end{aligned}$$

taking limit as $i \to \infty$ in the inequality, and using (2), we have

$$\lim_{j \to \infty} M(x_{n_j+1}, fx, t) = 1. \tag{6}$$

From (5) and (6), that is $fx = x$. □

In this section, coincidence point results for (φ)-weak contraction maps are obtained.

Theorem 3.2. *Let X be a nonempty set. Suppose that (X, M, Δ) be a complete fuzzy metric space, $T : X^m \to X$ be a (φ)-weak contraction with respect to self mapping f on X, $T(X^m) \subseteq f(X)$, f is continuous. Then f and T have a coincidence point in X provided that φ is a continuous mapping.*

Proof. By construct of the sequences $\{x_n^{(1)}\}_{n\geq 0}$, $\{x_n^{(2)}\}_{n\geq 0}$, $\ldots$, $\{x_n^{(m)}\}_{n\geq 0}$, from $T(X^m) \subseteq f(X)$ and let $\mathbf{x}_0 = (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(m)})$ in X^m. Choose a point $x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)}$ in X, such that $T\mathbf{x}_0 = T(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(m)}) = fx_1^{(j)}$ for some $j = 1, 2, \ldots, m$. Again, since $T(X^m) \subseteq f(X)$, we can choose a point $x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(m)}$ in X such that $T\mathbf{x}_1 = T(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)}) = fx_2^{(j)}$ for some $j = 1, 2, \ldots, m$. By this process indefinitely, for every $\{x_n^{(1)}\}_{n\geq 0}$, $\{x_n^{(2)}\}_{n\geq 0}$, $\ldots$, $\{x_n^{(m)}\}_{n\geq 0}$ such that $T\mathbf{x}_n = T(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)}) = fx_{n+1}^{(j)}$ for all n and some $j = 1, 2, \ldots, m$, suppose that $y_n^{(j)} = T\mathbf{x}_n = fx_{n+1}^{(j)}$. In order not to loss of generality, we can assume that $y_{n+1}^{(j)} \neq y_n^{(j)}$ for all $n \in \mathbb{N}$ and for all $j = (1, \ldots, m)$, otherwise f and T have a coincidence point and there is nothing to prove. In case $y_{n+1}^{(j)} \neq y_n^{(j)}$, using (1), we have

$$\begin{aligned}
\frac{1}{M(y_n^{(j)}, y_{n+1}^{(j)}, t)} - 1 &= \frac{1}{M(fx_{n+1}^{(j)}, fx_{n+2}^{(j)}, t)} - 1 \\
&= \frac{1}{M(T\mathbf{x}_n, T\mathbf{x}_{n+1}, t)} - 1 \\
&\leq \left(\frac{1}{M(fx_n^{(j)}, fx_{n+1}^{(j)}, t)} - 1\right) \\
&\quad - \varphi\left(\frac{1}{M(fx_n^{(j)}, fx_{n+1}^{(j)}, t)} - 1\right)
\end{aligned}$$

$$\leq \frac{1}{M(fx_n^{(j)},\ fx_{n+1}^{(j)},\ t)} - 1$$

$$= \frac{1}{M(y_{n-1}^{(j)},\ y_n^{(j)},\ t)} - 1, \tag{7}$$

which implies that $M(y_n^{(j)},\ y_{n+1}^{(j)},\ t) \geq M(y_{n-1}^{(j)},\ y_n^{(j)},\ t)$ for all n and therefore $M(y_{n-1}^{(j)},\ y_n^{(j)},\ t)$ is an increasing sequence and $M(y_{n-1}^{(j)},\ y_n^{(j)},\ t)$ in (0, 1]. Let $Q(t) = \lim_{n\to\infty} M(y_{n-1}^{(j)},\ y_n^{(j)},\ t)$. Next, we will show that $Q(t) = 1$ for all $t > 0$. On the other hand, there must exist some $t > 0$ such that $Q(t) < 1$. Letting $n \to \infty$ in (3), we get

$$\frac{1}{Q(t)} - 1 \leq \left(\frac{1}{Q(t)} - 1\right) - \varphi\left(\frac{1}{Q(t)} - 1\right),$$

which is a contradiction. Therefore $M(y_n^{(j)},\ y_{n+1}^{(j)},\ t) \to 1$ as $n \to \infty$. Note that, for each positive integer p,

$$M(y_n^{(j)}, y_{n+p}^{(j)}, t) \geq M(y_n^{(j)}, y_{n+1}^{(j)}, \frac{t}{p}) \Delta M(y_{n+1}^{(j)}, y_{n+2}^{(j)}, \frac{t}{p}) \Delta \ldots \Delta M(y_{n+p-1}^{(j)}, y_{n+p}^{(j)}, \frac{t}{p}).$$

This implies that

$$\lim_{n\to\infty} M(y_n^{(j)},\ y_{n+1}^{(j)},\ t) \geq 1 \,\Delta\, 1 \,\Delta \ldots \Delta\, 1 = 1.$$

Therefore, $\{y_n^{(j)}\}_{n \geq \mathbb{N}}$ are a Cauchy sequence, such that $y_n^{(j)} \to x^{(j)}$ as $n \to \infty$ for all j. Then we get a point $\mathbf{p} = (p_1,\ p_2,\ \ldots,\ p_m)$ in X^m such that $fp^{(j)} = x^{(j)}$. Next, we will show that $p^{(j)}$ is a coincidence point of f and T. To accomplish this; using (1), we get

$$\frac{1}{M(T\mathbf{p},\ fx_{n+1}^{(j)},\ t)} - 1 = \frac{1}{M(T\mathbf{p},\ T\mathbf{x}_n,\ t)} - 1$$

$$\leq \left(\frac{1}{M(fp^{(j)},\ fx_n^{(j)},\ t)} - 1\right) - \varphi\left(\frac{1}{M(fp^{(j)},\ fx_n^{(j)},\ t)} - 1\right)$$

for every $t > 0$. Letting limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} M(T\mathbf{p}, fx_{n+1}^{(j)}, t) = \lim_{n\to\infty} M(T\mathbf{p}, T\mathbf{x}_n, t)$$

$$= M(T\mathbf{p}, fp^{(j)}, t) = 1.$$

So $fp^j = T\mathbf{p}$. This completes the proof. □

Next, we present the example of Theorem 3.2.

Example 3.3. Let $X = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and Δ be a minimum norm. Let M be the fuzzy metric on X. Consider $T : X^m \to X$ and $f : X \to X$ given by

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \text{ for all } x, y \in X, t > 0.$$

Define function $\varphi : [0, \infty) \to [0, \infty)$ as $\varphi(t) = t$, and

$$T(x_1, x_2, \ldots, x_m) = \begin{cases} 0 & \text{if } x_1, x_2, \ldots, x_m \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}. \\ 1 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in \left\{\frac{1}{4}, \frac{1}{2}\right\}. \\ \frac{1}{2} & \text{if } x \in \left\{\frac{3}{4}, 1\right\} \end{cases}$$

Thus, we have the coincidence point in case $T\mathbf{x} = fx = 0$ and $T\mathbf{y} = fy = 0$. Thus, condition (1) is trivially satisfied if $x, y \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$. Suppose $x = 0$ and $y = 0$. Then, we have

$$\frac{1}{M(T\mathbf{x}, T\mathbf{y}, t)} - 1 \leq \left(\frac{1}{M(fx^{(i)}, fy^{(i)}, t)} - 1\right) - \varphi\left(\frac{1}{M(fx^{(i)}, fy^{(i)}, t)} - 1\right)$$

$$\frac{d(T\mathbf{x}, T\mathbf{y})}{t} \leq \frac{d(fx, fy)}{t} - \varphi\left(\frac{d(fx, fy)}{t}\right)$$

divided by t, thus

$$d(T\mathbf{x}, T\mathbf{y}) \leq d(fx, fy) - t\varphi\left(\frac{d(fx, fy)}{t}\right).$$

Thus all the conditions of Theorem 3.2 are satisfied. Moreover f and T have a coincidence point.

Theorem 3.4. *Let X be a nonempty set. Suppose that (X, M, Δ) is a complete fuzzy metric space. Consider the function $T : X^m \to X$ be a (φ)-weak contraction mapping with respect to self mapping f on X, $T(X^m) \subseteq f(X)$. If the functions T, f are commuting and f is continuous, then f and T have a common fixed point in X provided that φ is continuous mappings.*

Proof. By the proof of Theorem 3.2, we have a point $p \in X$ and $\mathbf{p} \in X^m$ such that $T\mathbf{p} = fp^j = q^j$ for $j = 1, 2, \ldots, m$, from T and f have a commutative, we obtain $fT(p_1, p_2, \ldots, p_m) = T(fp_1, fp_2, \ldots, fp_m)$. Obtain $fq^{(j)} = T\mathbf{q}$ for any $j = 1, 2, \ldots, m$, we show that $fq^{(j)} = q^{(j)}$. If $fq^{(j)} \neq q^{(j)}$, then

$$\begin{aligned}\frac{1}{M(fq^{(j)}, q^{(j)}, t)} - 1 &= \frac{1}{M(T\mathbf{q}, T\mathbf{p}, t)} - 1 \\ &\leq \left(\frac{1}{M(fq^{(j)}, fp^{(j)}, t)} - 1\right) \\ &\quad - \varphi\left(\frac{1}{M(fq^{(j)}, fp^{(j)}, t)} - 1\right) \\ &= \frac{1}{M(fq^{(j)}, q^{(j)}, t)} - 1 \\ &\quad - \varphi\left(\frac{1}{M(fq^{(j)}, q^{(j)}, t)} - 1\right)\end{aligned}$$

a contradiction which proves the result. Thus f and T have a common fixed point in X. This completes the proof. □

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