A numerical solution of fractional Black-Scholes equation

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Abstract

The Black-Scholes equation is the famous financial model that relates with an option. Nowadays, a fractional calculus is an excellent tool for solving problems in many vital fields. A fractional differential equation are widely used for many research with continuous and discrete approaches. In this research, the fractional Black-Scholes equation in financial problem is solved by using the numerical method. This equation is a fractional partial differential equation for the option price of a European call or European put under the Black-Scholes model. The implicit finite difference method and MLPG2 are used for discretizing the governing equation in time variable and option price, respectively. The time fractional derivative uses the Caputo partial derivative of fractional order α. The numerical examples for varieties of variables are also included.

Keywords: European option, fractional Black-Scholes equation, MLPG, moving kriging interpolation.

1. Introduction

The first idea of fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1965. Fractional calculus is a name for the theory of derivatives and integrals of arbitrary order. The famous definition of a fractional calculus are the Riemann-Liouville and Grunwald-Letnikov definition [1]. Caputo reformulated the definition of the Riemann-Liouvillein order to use integer order initial conditions to solve fractional differential equation [2]. Fractional differential equation have attracted much attention during the past few decades. This is the fact that fractional calculus supply an competent and excellent tool for the description of many important phenomena such as electromagnetic, physics, chemistry, biology, economy and many more.

Black-Scholes equation, which is proposed by Fisher Black and Myron Scholes [3], is the financial model that concern with option. An option is a contract between the seller and the buyer. It consists of a call option and a put option. Option valuation depends on the underlying asset price and time. The European option can only be exercised at the expiration date, but the American option can be exercised at any time before expiration date. The solution of Black-Scholes equation provides an option pricing formula for European option. The analytic solution is used in general case with basic assumption but it is not satisfied in some conditions. Some restrictions were appeared in the classical Black-Scholes equation that is the weaken of this model. Original assumptions were relieved by other models such as models with transaction cost [4-5], Jump-diffusion model [6], Stochastic volatility model [7] and Fractional Black-Scholes model [8-9].

Fractional Black-Scholes model is derived by many researchers. Some restrictions were appeared in the classical Black-Scholes equation that is the weaken of this model [10]. The Fractional Black-Scholes models are derived by substitute the standard Brownian motion with fractional Brownian motion.

In this paper, we propose a numerical method base on Meshless Local Petrov-Galerkin (MLPG) method to solve a fractional Black-Scholes equation. The MLPG is a truly meshless method, which involves not only a meshless interpolation for the trial functions, but also a meshless integration of the weak-form, [11]. MLPG2 is chosen for this research so the Kronecker delta is the test function. This method will avoid the domain integral in the weak-form.

2. Problem Formulation

The Black-Scholes equation is the outstanding financial equation that solve the European option pricing without a transaction cost. Moreover, underlying asset price distributed on the lognormal random walk, risk-free interest rate, no dividend and no arbitrate opportunity are fundamental assumption. The fractional Black-Scholes equation is following

$$\frac{\partial^{\alpha} u}{\partial \tau^{\alpha}} + r(\tau) s \frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2(s,\tau) s^2 \frac{\partial^2 u}{\partial s^2} - r(\tau) u = 0 \qquad (2.1)$$

$(s,\tau) \in \mathbb{R}^+ \times [0,T]$ with the terminal and boundary condition

$$u(s,T) = \max(s-E,0), s \in \mathbb{R}^+, u(0,\tau) = 0, \tau \in [0,T],$$

where $u(s,\tau)$ is the value of European call option at underlying asset price s at time τ, T *is* the expiration date, r is the risk-free interest rate, σ is the volatility of underlying asset price and E is the strike price. Basic definition of fractional calculus as following

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_{\mu}$, $\mu \geq -1$ is defined as [12],

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, (\alpha > 0)$$

$$J^0 f(t) = f(t).$$

For the The Riemann-Liouville fractional integral, we have:

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}$$

Definition 2. The fractional derivative of *f(t)* in the Caputo sense is defined as [13],

$$D_{\tau}^{\alpha} f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau,$$

for $m-1 < \alpha \leq m, m \in N, t > 0.$

For the Riemann-Liouville fractional integral and the Caputo fractional derivative, we have the following relation

$$J_{\tau}^{\alpha} D_{\tau}^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!}.$$

Definition 3. The Mittag-Leffler is defined as [14]

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, (\alpha \in C, \mathrm{Re}(\alpha) > 0).$$

From Eq.(2.1), when s goes to zero then degenerating will occur in approximation. We transform the Black-Scholes equation into a nondegenerate partial differential equation by using a logarithmic transformation $x = \ln s, t = \frac{1}{2}\sigma^2 (T-\tau),$ and define the computational domain for convenient in numerical experiments by

$\Omega = [x_{\min,} x_{\max}] \times [0,T]$,where
$x_{\min} = -\ln(4E), x_{\max} = \ln(4E)$, [15].

$$-\frac{1}{2}\sigma^2 \frac{\partial^{\alpha} u}{\partial t^{\alpha}} + r\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right) - ru = 0, \qquad (2.2)$$

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{2r}{\sigma^2}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - \frac{2r}{\sigma^2} u, \text{ where } k = \frac{2r}{\sigma^2}$$

Therefore $$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = (k-1)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - ku, \qquad (2.3)$$

$$u(x,0) = \max\left(e^x - E, 0\right), x \in (x_{\min}, x_{\max}),$$

$$u(x_{\min,}, t) = 0, u(x_{\max}, t) = e^{x_{\max}} - Ee^{-\int_0^t r(s)ds}, \ t \in [0,T].$$

3. Spatial Discretization

The MLPG method construct the local weak form over local subdomain, which is a small region taken for each node in global domain. Multiplying test function v_i into Eq.(2.3) and then integrate over subdomain (Ω_s^i) which is located inside the global domain (Ω) yields the following expression

$$\int_{\Omega_s^i} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} v_i d\Omega = \int_{\Omega_s^i} \left(\frac{\partial^2 u}{\partial x^2} + (k-1)\frac{\partial u}{\partial x} - ku\right) v_i d\Omega, \qquad (3.1)$$

Where v_i is a test function that make significant for each nodes. Rearrange Eq.(3.1), we have

$$\int_{\Omega_s^i} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} v_i d\Omega = \int_{\Omega_s^i} u,_{xx} v_i d\Omega + \int_{\Omega_s^i} (k-1) u,_x v_i d\Omega - \int_{\Omega_s^i} kuv_i d\Omega, \qquad (3.2)$$

Where $u,_{xx} = \frac{\partial^2 u}{\partial x^2}, u,_x = \frac{\partial u}{\partial x}$. Substituting trial function $u^h(x,t) = \sum_{j=1}^{N} \phi_j(x) \hat{u}_j(t)$ into u in Eq.(3.2)

$$\int \sum_{j=1}^{N} \phi_j(x) v_i(x) \frac{\partial^{\alpha} \hat{u}_j(t)}{\partial t^{\alpha}} d\Omega = \int_{\Omega_s^i} \sum_{j=1}^{N} \phi_{j,xx}(x) \ v_i(x) \hat{u}_j(t) d\Omega$$

$$+(k-1)\int_{\Omega_s^i}\sum_{j=1}^{N}\phi_{j,x}(x)v_i(x)\hat{u}_j(t)d\Omega$$

$$-k\int_{\Omega_s^i}\sum_{j=1}^{N}\phi_j(x)v_i(x)\hat{u}_j(t)d\Omega \tag{3.3}$$

where N is the number of nodes surrounding point x which has the effect on $u(x)$ and $\hat{u}_j$ is value of option at time t. The shape function, ϕ_j, is constructed by moving kriging interpolation which has the Kronecker delta property, thereby enhancing the arrangement nodal shape construction accuracy. Rearrange Eq.(3.3) yields the following result

$$\sum_{j=1}^{N}\int_{\Omega_s^i}\phi_j(x)v_i(x)\left(\frac{\partial^\alpha\hat{u}_j(t)}{\partial t^\alpha}\right)d\Omega=\sum_{j=1}^{N}\int_{\Omega_s^i}\phi_{j,xx}(x)v_i(x)\hat{u}_j(t)d\Omega$$

$$+(k-1)\sum_{j=1}^{N}\int_{\Omega_s^i}\phi_{j,x}(x)v_i(x)\hat{u}_j(t)d\Omega$$

$$-k\sum_{j=1}^{N}\int_{\Omega_s^i}\phi_j(x)v_i(x)\hat{u}_j(t)d\Omega \tag{3.4}$$

This research use MLPG2 then the test function is chosen by Kronecker delta function,

$$v_i(x)=\begin{cases}0, & x\neq x_i\\ 1, & x=x_i\end{cases},\ i=1,2,\ldots,N.$$

The test function will define significance for each node in subdomain. In this case, substituting test function $v_i(x)$ to Eq.(3.4) and then integrate over subdomain Ω_s^i yields the following result

$$\sum_{j=1}^{N}\phi_j(x_i)\frac{d^\alpha\hat{u}_j(t)}{dt^\alpha}=\sum_{j=1}^{N}[\phi_{j,xx}(x_i)+(k-1)\phi_{j,x}(x_i)$$

$$-k\phi_j(x_i)]\hat{u}_j(t) \tag{3.5}$$

Eq.(3.5) can be written in the matrix form as following

$$A\frac{d^\alpha U}{dt^\alpha}=BU, \tag{3.6}$$

where $A=[A_{ij}]_{N\times N}, A_{ij}=\phi_j(x_i)$,

$B=[B_{ij}]_{N\times N}, B_{ij}=\phi_{j,xx}(x_i)+(k-1)\phi_{j,x}(x_i)-k\phi_j(x_i)$,

$U=[\hat{u}_1\ \hat{u}_2\ \hat{u}_3\ldots\hat{u}_N]^T$

Since the shape function that is constructed by the moving kriging interpolation satisfy the Kronecker delta property, A is the identity matrix. Therefore, Eq.(3.6) can be written as

$$\frac{d^\alpha U}{dt^\alpha}=BU \tag{3.7}$$

4. Temporal Discretization

The numerical solution of European option use the implicit finite difference method. By a finite approximation made for the time fractional derivative with notation $\frac{\partial^\alpha u(x_i,t_n)}{\partial t^\alpha}$ that approximates the exact solution $u(x_i,t_n)$ at time level n, we restrict attention to the finite space domain $x_{\min}<x<x_{\max}$ with $0<\alpha<1$. The time fractional derivative use the implicit finite difference [16], defined by

$$\frac{d^\alpha U}{dt^\alpha}=\sigma_{\alpha,\Delta t}\sum_{j=1}^{n}\left(U^{n-j+1}-U^{n-j}\right)+O(\Delta t) \tag{4.1}$$

where $\sigma_{\alpha,\Delta t}=\frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\frac{1}{(\Delta t)^\alpha}$.

Hence, $\frac{d^\alpha U}{dt^\alpha}=D_t^{(\alpha)}U_i^n+O(\Delta t)$.

The first-order approximation method for the computation of Caputo's fractional derivative is given by

$$D_t^{(\alpha)}U^n=\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left(U^{n-j+1}-U^{n-j}\right) \tag{4.2}$$

where $\omega_j^{(\alpha)}=j^{1-\alpha}-(j-1)^{1-\alpha}$.

Consider the Eq. (4.2) and substitute time fractional derivative that following

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left(U^{n-j+1}-U^{n-j}\right)+O(\Delta t)=BU^n,$$

$$\sigma_{\alpha,k}\omega_1^{(\alpha)}\left(U^n-U^{n-1}\right)=-\sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(U^{n-j+1}-U^{n-j}\right)+BU^n,$$

We consider the first case for $n=1$,

Case $n=1$

$$\sigma_{\alpha,k}\omega_1^{(\alpha)}\left(U^1-U^0\right)=BU^1,$$
$$(\sigma_{\alpha,k}\omega_1^{(\alpha)}I-B)U^1=\sigma_{\alpha,k}\omega_1^{(\alpha)}U^0,$$

Case $n\geq 2$

$$\left(\sigma_{\alpha,k}\omega_1^{(\alpha)}I-B\right)U^n=\sigma_{\alpha,k}\omega_1^{(\alpha)}U^{n-1}$$
$$-\sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(U^{n-j+1}-U^{n-j}\right).$$

5. Numerical Examples

In this section, we are going to present various numerical results to evaluate proposed meshless approaches. Using the MLPG2 method, the resulting problems for European call options are solved via implicit finite difference method.

The European call option can be modeled by fractional Black-Scholes PDE as following : [17]

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\left(k-1\right)\frac{\partial u}{\partial x}-ku,\ \ 0<\alpha<1. \tag{5.1}$$

where $k=\dfrac{2r}{\sigma^2}$.

with initial condition given by

$$u\left(x,0\right)=\max\left(e^x-E,0\right),\ \ x\in\mathbb{R}^+. \tag{5.2}$$

The analytical solution for the European call option is

$$\begin{aligned}u\left(x,t\right)&=\max\left(e^x,0\right)(1-E_\alpha(-kt^\alpha))\\&+\max\left(e^x-1,0\right)E_\alpha(-kt^\alpha)\end{aligned} \tag{5.3}$$

where $E_\alpha(-kt^\alpha)=\sum_{n=0}^{\infty}\dfrac{(-kt^\alpha)^n}{\Gamma(n\alpha+1)}$

Example 1. We consider the fractional Black-Scholes equation in Eq.(5.1). The numerical simulation was done for European call option with parameters as following:

Case 1. For $\sigma=0.2,r=0.04,\alpha=0.5,T=2,k=2$. In this case , we get the exact solution as following : $u\left(x,t\right)=\max\left(e^x,0\right)(1-e^{-2t})+\max\left(e^x-1,0\right)e^{-2t}$,

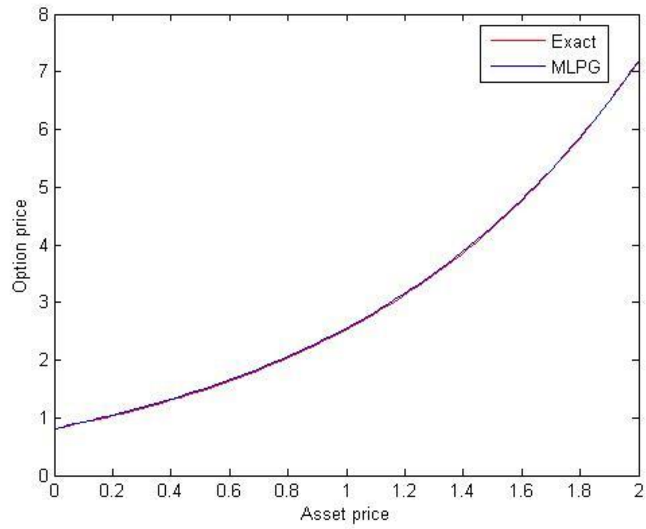

Figure 1. The approximate solution compare with the exact solution for $\sigma=0.2,r=0.04,$ $\alpha=0.5,k=2,t=T.$

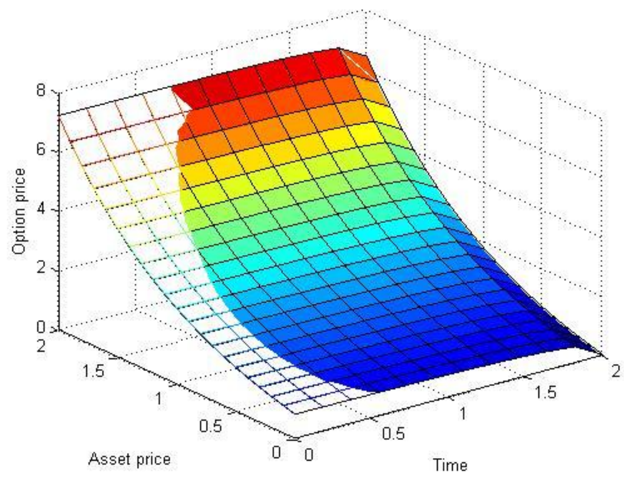

Figure 2. The approximate solution compare with the exact solution for $\sigma=0.2,r=0.04,\alpha=0.5,k=2,$ $0\leq t\leq T.$

Case 2. For $\sigma=0.2,r=0.01,\alpha=0.99,T=4,k=5$ In this case, we get the exact solution as following : $u\left(x,t\right)=\max\left(e^x,0\right)(1-e^{-5t})+\max\left(e^x-1,0\right)e^{-5t}$,

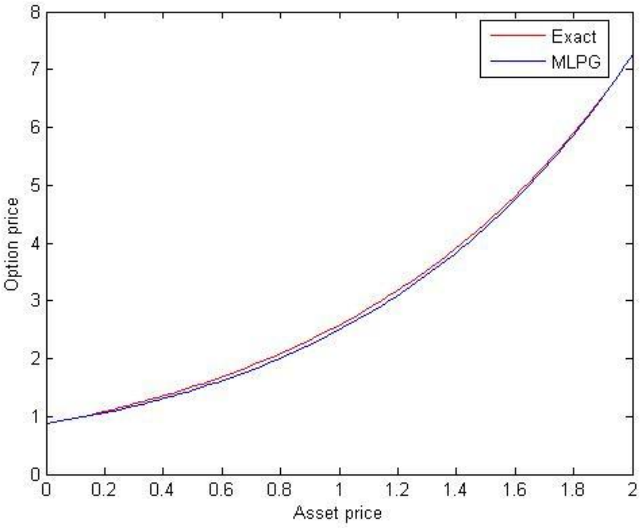

Figure 3. The approximate solution compare with the exact solution for $\sigma = 0.2, r = 0.01, \alpha = 0.99,$

$k = 5, t = T.$

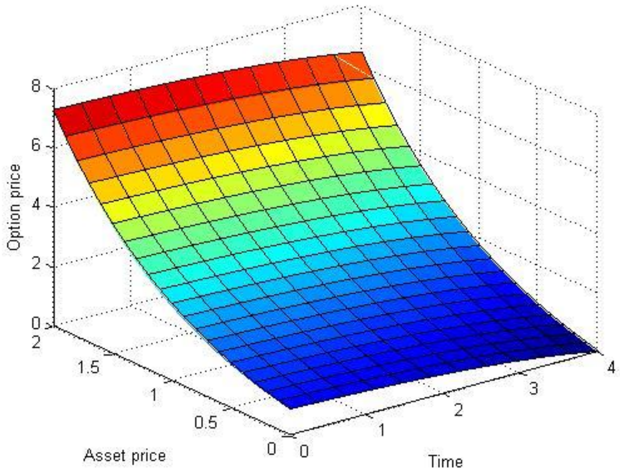

Figure 4. The approximate solution compare with the exact solution for $\sigma = 0.2, r = 0.01, \alpha = 0.99, k = 5,$

$0 \le t \le T.$

Case 3. For $\sigma = 0.1, r = 0.06, \alpha = 0.99, T = 1, k = 12$. In this case , we get the exact solution as following :

$$u(x,t) = \max\left(e^{x}, 0\right)(1 - e^{-12t}) + \max\left(e^{x} - 1, 0\right)e^{-12t},$$

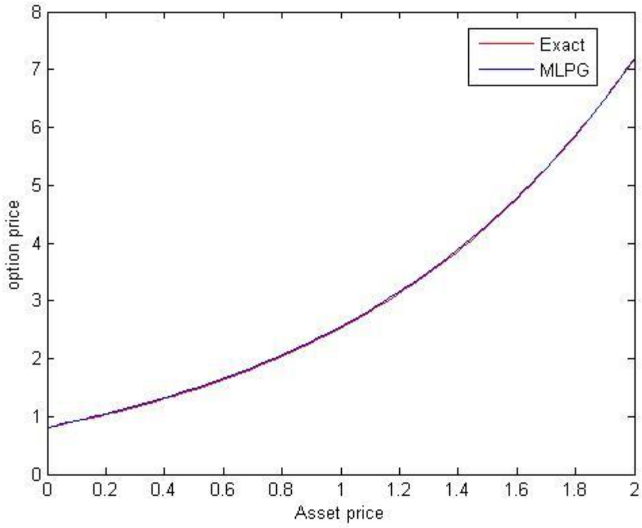

Figure 5. The approximate solution compare with the exact solution for $\sigma = 0.1, r = 0.06, \alpha = 0.99,$

$k = 12, t = T.$

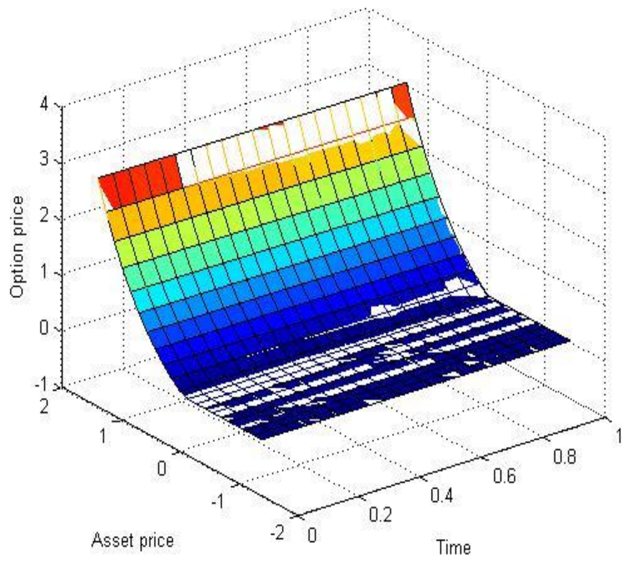

Figure 6. The approximate solution compare with the exact solution for $\sigma = 0.1, r = 0.06, \alpha = 0.99, k = 12$

$0 \le t \le T.$

Case 4. For $\sigma = 0.4(2 + \sin x), r = 0.06,$

$$\alpha = 0.99, T = 1, k = \frac{2r}{\sigma^2}.$$

In this case , the exact solution is unknown.

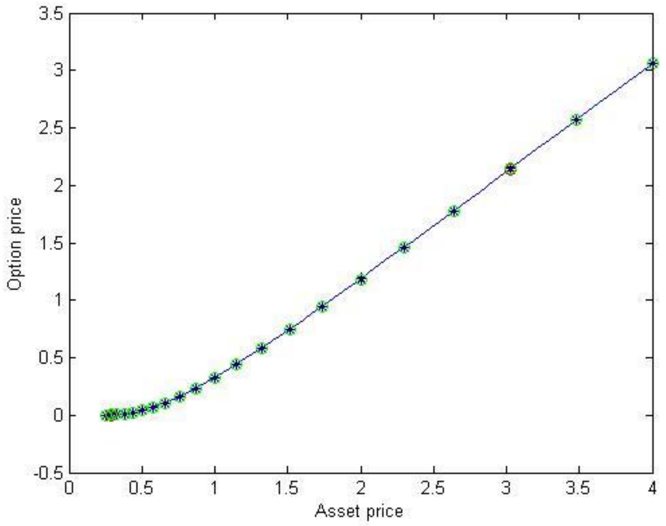

Figure 7. The comparison of the approximate solutions of the fractional and standard Black-Scholes equation

for $r = 0.06, \alpha = 0.99, k = \dfrac{2r}{\sigma^2}, t = T.$

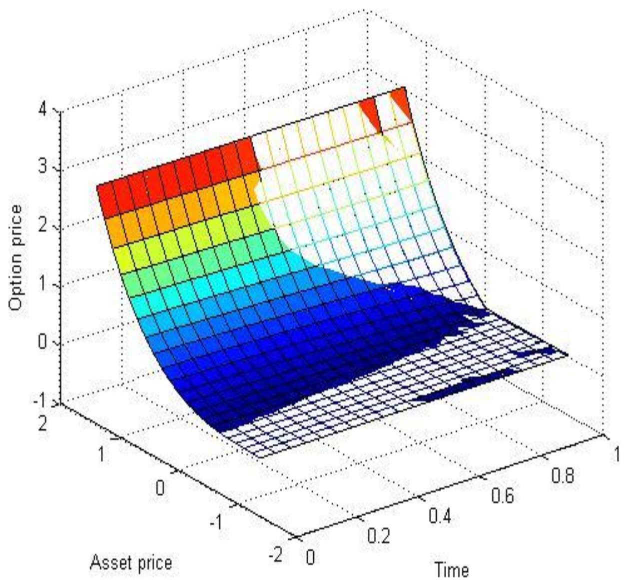

Figure 8. The comparison of the approximate solutions of the fractional and standard Black-Scholes equation

$r = 0.06, \alpha = 0.99, k = \dfrac{2r}{\sigma^2}, 0 \le t \le T.$

6. Conclusion

In this paper, the fractional Black-Scholes equations are solved by the implicit finite difference method and MLPG2 for discretizing in time variable and option price, respectively. The Caputo partial derivative of fractional order α are used for numerical scheme.

The numerical results are presented in four cases. Case 1 and 2 presents numerical results for varieties of parameters and $\alpha = 0.5, 0.99.$ In figure 1.and 3., we found that the last time have a little difference value of option. In figure 2., the value of option will only differ for initial time and case $\alpha = 0.99,$ the value of option have no difference from exact solution for all time. Case 3 present various parameters and final case show comparison of approximation solutions of the fractional Black-Scholes equation and standard Black-Scholes equation.

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